Non-Life Insurance: Mathematics and Statistics

Solution sheet 7

Solution 7.1 Re-Insurance Covers and Leverage Effect

(a) Under the assumption that $\mathbb{P}[Y > d] > 0$ and that $\mathbb{E}[Y|Y > d]$ exists, we can generally write

$$\begin{aligned}\mathbb{E}[(Y-d)_+] &= \mathbb{E}[(Y-d)1_{\{Y>d\}}] = \mathbb{E}[Y1_{\{Y>d\}}] - \mathbb{E}[d1_{\{Y>d\}}] \\ &= \frac{\mathbb{E}[Y1_{\{Y>d\}}]}{\mathbb{P}[Y>d]}\mathbb{P}[Y>d] - d\,\mathbb{P}[Y>d] = \mathbb{P}[Y>d](\mathbb{E}[Y|Y>d]-d),\end{aligned}$$

see also formula (3.11) of the lecture notes (version of March 20, 2019). Now we explicitly use that a gamma distribution with shape parameter equal to 1 is an exponential distribution. The characteristic property of an exponential distribution is the so-called memorylessness property

$$\mathbb{P}[Y > t+s|Y>t] = \mathbb{P}[Y>s],$$

for all $t, s > 0$. In particular, this property leads to (see below for the calculation)

$$\mathbb{E}[Y|Y>d] = \mathbb{E}[Y] + d, \tag{1}$$

which, in turn, implies for our loss $Y \sim \Gamma(1, \frac{1}{400})$ that

$$\mathbb{E}[(Y-d)_+] = \mathbb{P}[Y>d]\mathbb{E}[Y].$$

We check equation (1). Indeed, we have

$$\begin{aligned}\mathbb{E}[Y|Y>d] &= \frac{\mathbb{E}[Y1_{\{Y>d\}}]}{\mathbb{P}[Y>d]} = \frac{1}{\mathbb{P}[Y>d]}\int_0^\infty y1_{\{y>d\}}\frac{1}{400}\exp\left\{-\frac{y}{400}\right\}dy \\ &= \frac{1}{\exp\left\{-\frac{d}{400}\right\}}\int_d^\infty y\frac{1}{400}\exp\left\{-\frac{y}{400}\right\}dy \\ &= \exp\left\{\frac{d}{400}\right\}\int_0^\infty (u+d)\frac{1}{400}\exp\left\{-\frac{u}{400}\right\}\exp\left\{-\frac{d}{400}\right\}du \\ &= \int_0^\infty u\frac{1}{400}\exp\left\{-\frac{u}{400}\right\}du + d\int_0^\infty \frac{1}{400}\exp\left\{-\frac{u}{400}\right\}du \\ &= \mathbb{E}[Y]+d,\end{aligned}$$

where in the fourth equality we used the substitution $u = y - d$.

(b) By looking at the graphs in Figure 1 on the exercise sheet, we find the following re-insurance covers:

(i) $(Y-200)_+$,

(ii) $\min\{Y, 400\}$,

(iii) $\min\{Y, 200\} + (Y-400)_+$.

(c) (i) Using part (a), we get

$$\mathbb{E}[(Y-200)_+] = \mathbb{P}[Y>200]\mathbb{E}[Y] = \exp\left\{-\frac{200}{400}\right\}400 = \frac{400}{\sqrt{\exp\{1\}}} \approx 243.$$

(ii) First, we write

$$\begin{aligned}\mathbb{E}[\min\{Y,400\}] &= \mathbb{E}[\min\{Y,400\}1_{\{Y\le 400\}}] + \mathbb{E}[\min\{Y,400\}1_{\{Y>400\}}]\\ &= \mathbb{E}[Y1_{\{Y\le 400\}}] + \mathbb{E}[400\cdot 1_{\{Y>400\}}]\\ &= \mathbb{E}[Y] - \mathbb{E}[Y1_{\{Y>400\}}] + \mathbb{E}[400\cdot 1_{\{Y>400\}}]\\ &= \mathbb{E}[Y] - \mathbb{E}[(Y-400)1_{\{Y>400\}}]\\ &= \mathbb{E}[Y] - \mathbb{E}[(Y-400)_+],\end{aligned}$$

which holds true as $\mathbb{E}[Y]$ exists, see also page 89 the lecture notes (version of March 20, 2019). Using part (a), we then get

$$\begin{aligned}\mathbb{E}[\min\{Y,400\}] &= \mathbb{E}[Y] - \mathbb{P}[Y>400]\mathbb{E}[Y] = \mathbb{E}[Y](1-\mathbb{P}[Y>400])\\ &= 400\left(1-\exp\left\{-\frac{400}{400}\right\}\right) = 400(1-\exp\{-1\}) \approx 253.\end{aligned}$$

(iii) Using the above calculation in (ii) as well as part (a), we have

$$\begin{aligned}\mathbb{E}[\min\{Y,200\}+(Y-400)_+] &= \mathbb{E}[Y] - \mathbb{E}[(Y-200)_+] + \mathbb{E}[(Y-400)_+]\\ &= \mathbb{E}[Y] - \mathbb{P}[Y>200]\mathbb{E}[Y] + \mathbb{P}[Y>400]\mathbb{E}[Y]\\ &= \mathbb{E}[Y](1-\mathbb{P}[Y>200]+\mathbb{P}[Y>400])\\ &= 400\left(1-\exp\left\{-\frac{1}{2}\right\}+\exp\{-1\}\right)\\ &\approx 305.\end{aligned}$$

(d) As $Y_0 \sim \Gamma\left(1, \frac{1}{400}\right)$, formula (3.5) of the lecture notes (version of March 20, 2019) implies

$$Y_1 \overset{(d)}{=} (1+i)Y_0 \sim \Gamma\left(1, \frac{1}{400(1+i)}\right).$$

Using part (a), we get

$$\mathbb{E}[(Y_1-d)_+] = \mathbb{P}[Y_1>d]\mathbb{E}[Y_1] = \exp\left\{-\frac{d}{400(1+i)}\right\}400(1+i)$$

and

$$\mathbb{E}[(Y_0-d)_+] = \mathbb{P}[Y_0>d]\mathbb{E}[Y_0] = \exp\left\{-\frac{d}{400}\right\}400,$$

which leads to

$$\frac{\mathbb{E}[(Y_1-d)_+]}{(1+i)\mathbb{E}[(Y_0-d)_+]} = \frac{\exp\left\{-\frac{d}{400(1+i)}\right\}}{\exp\left\{-\frac{d}{400}\right\}} = \exp\left\{\frac{d}{400}\left(1-\frac{1}{1+i}\right)\right\} > 1,$$

since $i>0$. We conclude that

$$\mathbb{E}[(Y_1-d)_+] > (1+i)\mathbb{E}[(Y_0-d)_+].$$

The reason for this (strict) inequality, which is called leverage effect, is that not only the claim sizes are growing under inflation, but also the number of claims that exceed threshold d increases under inflation, as we do not adapt threshold d to inflation.

Solution 7.2 Inflation and Deductible

Let Y be a random variable following a Pareto distribution with threshold $\theta > 0$ and tail index $\alpha > 1$. Since the insurance company only has to pay the part that exceeds the deductible θ, this year's average claim payment z is

$$z = \mathbb{E}\left[(Y-\theta)_+\right] = \mathbb{E}[Y] - \theta = \frac{\alpha}{\alpha-1}\theta - \theta = \frac{1}{\alpha-1}\theta.$$

For the total claim size $\widetilde{Y}$ of a claim next year we have

$$\widetilde{Y} \overset{(d)}{=} (1+r)Y \sim \text{Pareto}([1+r]\theta, \alpha).$$

Thus, the mean excess function $e_{\widetilde{Y}}(u)$ of $\widetilde{Y}$ above $u > (1+r)\theta$ is given by

$$e_{\widetilde{Y}}(u) = \frac{1}{\alpha-1}u,$$

see also Exercise 5.4. Let $\rho\theta$ for some $\rho > 0$ denote the increase of the deductible that is needed such that the average claim payment remains unchanged. With the new deductible $(1+\rho)\theta$, next year's average claim payment is given by

$$\widetilde{z} = \mathbb{E}\left[\left(\widetilde{Y} - [1+\rho]\theta\right)_+\right].$$

The goal is to find $\rho > 0$ such that $z = \widetilde{z}$. Assuming $\rho \leq r$, we have

$$\begin{aligned}\widetilde{z} &= \mathbb{E}\left[\left(\widetilde{Y} - [1+\rho]\theta\right)_+\right] \geq \mathbb{E}\left[\left(\widetilde{Y} - [1+r]\theta\right)_+\right] = \mathbb{E}\left[([1+r]Y - [1+r]\theta)_+\right] \\ &= (1+r)\,\mathbb{E}\left[(Y-\theta)_+\right] = (1+r)\,z > z,\end{aligned}$$

i.e. for $\rho \leq r$ it is not possible to get $z = \widetilde{z}$. Hence, we can deduce that $\rho > r$, i.e. the percentage increase in the deductible has to be bigger than the inflation. Assuming $\rho > r$, we can calculate

$$\begin{aligned}\widetilde{z} &= \mathbb{E}\left[\left(\widetilde{Y} - [1+\rho]\theta\right)\cdot 1_{\{\widetilde{Y}-(1+\rho)\theta>0\}}\right] = \mathbb{E}\left[\widetilde{Y} - (1+\rho)\theta \,\middle|\, \widetilde{Y} > (1+\rho)\theta\right]\cdot \mathbb{P}\left[\widetilde{Y} > (1+\rho)\theta\right] \\ &= e_{\widetilde{Y}}([1+\rho]\theta)\cdot \mathbb{P}\left[\widetilde{Y} > (1+\rho)\theta\right] = \frac{1}{\alpha-1}(1+\rho)\theta \cdot \left[\frac{(1+\rho)\theta}{(1+r)\theta}\right]^{-\alpha} \\ &= \frac{1}{\alpha-1}\theta(1+r)^{\alpha}(1+\rho)^{-\alpha+1} = z\cdot(1+r)^{\alpha}(1+\rho)^{-\alpha+1}.\end{aligned}$$

We have

$$z = \widetilde{z} \iff (1+r)^{\alpha}(1+\rho)^{-\alpha+1} = 1 \iff \rho = (1+r)^{\frac{\alpha}{\alpha-1}} - 1 > r.$$

We conclude that if we want the average claim payment to remain unchanged, we have to increase the deductible θ by the amount

$$\theta\left[(1+r)^{\frac{\alpha}{\alpha-1}} - 1\right].$$

Solution 7.3 Normal Approximation

As $Y \sim \Gamma(\gamma = 100, c = \frac{1}{10})$, we have

$$\begin{aligned}\mathbb{E}[Y] &= \frac{\gamma}{c} = \frac{100}{1/10} = 1'000 \quad \text{and} \\ \mathbb{E}[Y^2] &= \frac{\gamma(\gamma+1)}{c^2} = \frac{100\cdot 101}{1/100} = 1'010'000.\end{aligned}$$

For the total claim amount S we can use Proposition 2.11 of the lecture notes (version of March 20, 2019) to get

$$\mathbb{E}[S] = \lambda v \mathbb{E}[Y] = 1'000 \cdot 1'000 = 1'000'000 \quad \text{and}$$
$$\text{Var}(S) = \lambda v \mathbb{E}[Y^2] = 1'000 \cdot 1'010'000 = 1'010'000'000.$$

Let F_S denote the distribution function of S. Then, since F_S is continuous and strictly increasing (above the level $\exp\{-\lambda v\} = \mathbb{P}[S = 0]$), the quantiles $q_{0.95}$ and $q_{0.99}$ can be calculated as

$$q_{0.95} = F_S^{-1}(0.95) \qquad \text{and} \qquad q_{0.99} = F_S^{-1}(0.99).$$

According to Section 4.1.1 of the lecture notes (version of March 20, 2019), the normal approximation is given by

$$F_S(x) \approx \Phi\left(\frac{x - \lambda v \mathbb{E}[Y]}{\sqrt{\lambda v \mathbb{E}[Y^2]}}\right),$$

for all $x \in \mathbb{R}$, where Φ is the standard Gaussian distribution function. For all $\alpha \in (0, 1)$ we then have

$$F_S^{-1}(\alpha) = \lambda v \mathbb{E}[Y] + \sqrt{\lambda v \mathbb{E}[Y^2]} \cdot \Phi^{-1}(\alpha) = 1'000 \cdot 1'000 + \sqrt{1'000 \cdot 1'010'000} \cdot \Phi^{-1}(\alpha)$$
$$\approx 1'000'000 + 31'780.5 \cdot \Phi^{-1}(\alpha).$$

In particular, we get

$$q_{0.95} = F_S^{-1}(0.95) \approx 1'000'000 + 31'780.5 \cdot \Phi^{-1}(0.95) \approx 1'000'000 + 31'780.5 \cdot 1.645 \approx 1'052'274$$

and

$$q_{0.99} = F_S^{-1}(0.99) \approx 1'000'000 + 31'780.5 \cdot \Phi^{-1}(0.99) \approx 1'000'000 + 31'780.5 \cdot 2.326 \approx 1'073'932.$$

Note that the normal approximation also allows for negative claims S, which under our model assumptions is excluded. The probability for negative claims S in the normal approximation can be calculated as

$$F_S(0) \approx \Phi\left(\frac{0 - \lambda v \mathbb{E}[Y]}{\sqrt{\lambda v \mathbb{E}[Y^2]}}\right) \approx \Phi\left(-\frac{1'000'000}{31'780.5}\right) \approx \Phi(-31.5) \approx 1.27 \cdot 10^{-217},$$

which of course is positive, but very close to 0.

Solution 7.4 Translated Gamma and Translated Log-Normal Approximation

As $Y \sim \Gamma(\gamma = 100, c = \frac{1}{10})$, we have

$$\mathbb{E}[Y] = \frac{\gamma}{c} = \frac{100}{1/10} = 1'000,$$
$$\mathbb{E}[Y^2] = \frac{\gamma(\gamma+1)}{c^2} = \frac{100 \cdot 101}{1/100} = 1'010'000 \quad \text{and}$$
$$\mathbb{E}[Y^3] = \frac{\gamma(\gamma+1)(\gamma+2)}{c^3} = \frac{100 \cdot 101 \cdot 102}{1/1'000} = 1'030'200'000.$$

Let M_Y denote the moment generating function of Y. According to formula (1.3) of the lecture notes (version of March 20, 2019), we have

$$M_Y'''(0) = \frac{d^3}{dr^3} M_Y(r)\bigg|_{r=0} = \mathbb{E}[Y^3].$$

For the total claim amount S we can use Proposition 2.11 of the lecture notes (version of March 20, 2019) to get

$$\begin{aligned} \mathbb{E}[S] &= \lambda v \mathbb{E}[Y] = 1'000 \cdot 1'000 = 1'000'000, \\ \mathrm{Var}(S) &= \lambda v \mathbb{E}[Y^2] = 1'000 \cdot 1'010'000 = 1'010'000'000 \quad \text{and} \\ M_S(r) &= \exp\{\lambda v [M_Y(r) - 1]\}, \end{aligned}$$

where M_S denotes the moment generating function of S. In order to get the skewness ς_S of S, we can use the third equation given in formulas (1.5) of the lecture notes (version of March 20, 2019):

$$\varsigma_S \cdot \mathrm{Var}(S)^{3/2} = \frac{d^3}{dr^3} \log M_S(r)\Big|_{r=0} = \lambda v \frac{d^3}{dr^3}[M_Y(r) - 1]\Big|_{r=0} = \lambda v M_Y'''(0) = \lambda v \mathbb{E}[Y^3],$$

from which we can conclude that

$$\varsigma_S = \frac{\lambda v \mathbb{E}[Y^3]}{(\lambda v \mathbb{E}[Y^2])^{3/2}} = \frac{\mathbb{E}[Y^3]}{\sqrt{\lambda v}\,\mathbb{E}[Y^2]^{3/2}} = \frac{1'030'200'000}{\sqrt{1'000}(1'010'000)^{3/2}} \approx 0.0321.$$

Let F_S denote the distribution function of S. Then, since F_S is continuous and strictly increasing (above the level $\exp\{-\lambda v\} = \mathbb{P}[S = 0]$), the quantiles $q_{0.95}$ and $q_{0.99}$ can be calculated as

$$q_{0.95} = F_S^{-1}(0.95) \qquad \text{and} \qquad q_{0.99} = F_S^{-1}(0.99).$$

(a) According to Section 4.1.2 of the lecture notes (version of March 20, 2019), in the translated gamma approximation we model S by the random variable

$$X = k + Z,$$

where $k \in \mathbb{R}$ and $Z \sim \Gamma(\widetilde{\gamma}, \widetilde{c})$. The three parameters $k, \widetilde{\gamma}$ and $\widetilde{c}$ can be determined by solving the equations

$$\mathbb{E}[X] = \mathbb{E}[S], \qquad \mathrm{Var}(X) = \mathrm{Var}(S) \qquad \text{and} \qquad \varsigma_X = \varsigma_S, \tag{2}$$

where ς_X is the skewness parameter of X. Since $Z \sim \Gamma(\widetilde{\gamma}, \widetilde{c})$, we can use the results given in Section 3.2.1 of the lecture notes (version of March 20, 2019) to calculate

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[k + Z] = k + \mathbb{E}[Z] = k + \frac{\widetilde{\gamma}}{\widetilde{c}}, \\ \mathrm{Var}(X) &= \mathrm{Var}(k + Z) = \mathrm{Var}(Z) = \frac{\widetilde{\gamma}}{\widetilde{c}^2} \quad \text{and} \\ \varsigma_X &= \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^3\right]}{\mathrm{Var}(X)^{3/2}} = \frac{\mathbb{E}\left[(k + Z - \mathbb{E}[k + Z])^3\right]}{\mathrm{Var}(k + Z)^{3/2}} = \frac{\mathbb{E}\left[(Z - \mathbb{E}[Z])^3\right]}{\mathrm{Var}(Z)^{3/2}} = \varsigma_Z = \frac{2}{\sqrt{\widetilde{\gamma}}}. \end{aligned}$$

Using the equations given in (2), we get

$$\begin{aligned} \frac{2}{\sqrt{\widetilde{\gamma}}} = \varsigma_S \quad &\Longleftrightarrow \quad \widetilde{\gamma} = \frac{4}{\varsigma_S^2} \approx 3'883, \\ \frac{\widetilde{\gamma}}{\widetilde{c}^2} = \mathrm{Var}(S) \quad &\Longleftrightarrow \quad \widetilde{c} = \sqrt{\frac{\widetilde{\gamma}}{\mathrm{Var}(S)}} \approx 0.002 \quad \text{and} \\ k + \frac{\widetilde{\gamma}}{\widetilde{c}} = \mathbb{E}[S] \quad &\Longleftrightarrow \quad k = \mathbb{E}[S] - \frac{\widetilde{\gamma}}{\widetilde{c}} \approx -980'392. \end{aligned}$$

If we write F_Z for the distribution function of $Z \sim \Gamma(\widetilde{\gamma} \approx 3'883, \widetilde{c} \approx 0.002)$, we get using the translated gamma approximation

$$F_S(x) = \mathbb{P}[S \leq x] \approx \mathbb{P}[X \leq x] = \mathbb{P}[k + Z \leq x] = \mathbb{P}[Z \leq x - k] = F_Z(x - k),$$

for all $x \in \mathbb{R}$. Now, for all $\alpha \in (0,1)$, we have

$$F_S^{-1}(\alpha) \approx k + F_Z^{-1}(\alpha).$$

In particular, we get

$$q_{0.95} = F_S^{-1}(0.95) \approx k + F_Z^{-1}(0.95) \approx -980'392 + 2'032'955 = 1'052'563$$

and

$$q_{0.99} = F_S^{-1}(0.99) \approx k + F_Z^{-1}(0.99) \approx -980'392 + 2'055'074 = 1'074'682.$$

Note that since $k < 0$, the translated gamma approximation in this example also allows for negative claims S, which under our model assumptions is excluded. The probability for negative claims S can be calculated as

$$F_S(0) \approx F_Z(0-k) \approx F_Z(980'392) \approx 4.87 \cdot 10^{-320},$$

which is basically 0.

(b) According to Section 4.1.2 of the lecture notes (version of March 20, 2019), in the translated log-normal approximation we model S by the random variable

$$X = k + Z,$$

where $k \in \mathbb{R}$ and $Z \sim \mathrm{LN}(\mu, \sigma^2)$. Similarly as in part (b), the three parameters k, μ and σ^2 can be determined by solving the equations

$$\mathbb{E}[X] = \mathbb{E}[S], \qquad \mathrm{Var}(X) = \mathrm{Var}(S) \qquad \text{and} \qquad \varsigma_X = \varsigma_S. \tag{3}$$

Since $Z \sim \mathrm{LN}(\mu, \sigma^2)$, we can use the results given in Section 3.2.3 of the lecture notes (version of March 20, 2019) to calculate

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[k+Z] = k + \mathbb{E}[Z] = k + \exp\left\{\mu + \sigma^2/2\right\}, \\ \mathrm{Var}(X) &= \mathrm{Var}(k+Z) = \mathrm{Var}(Z) = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right) \quad \text{and} \\ \varsigma_X &= \varsigma_Z = \left(\exp\left\{\sigma^2\right\} + 2\right)\left(\exp\left\{\sigma^2\right\} - 1\right)^{1/2}. \end{aligned}$$

Using the third equation in (3), we get

$$\left(\exp\left\{\sigma^2\right\} + 2\right)\left(\exp\left\{\sigma^2\right\} - 1\right)^{1/2} = \varsigma_S \approx 0.0321 \quad \Longleftrightarrow \quad \sigma^2 \approx 0.00011444,$$

which was found using a root search algorithm. Using the second equation in (3), we get

$$\exp\left\{2\mu + \sigma^2\right\}\left(\exp\left\{\sigma^2\right\} - 1\right) = \mathrm{Var}(S) \iff \mu = \frac{1}{2}\left(\log\left[\left(\exp\left\{\sigma^2\right\} - 1\right)^{-1}\mathrm{Var}(S)\right] - \sigma^2\right),$$

which implies

$$\mu \approx 14.90425.$$

Finally, using the first equation in (3), we get

$$k + \exp\left\{\mu + \sigma^2/2\right\} = \mathbb{E}[S] \quad \Longleftrightarrow \quad k = \mathbb{E}[S] - \exp\left\{\mu + \sigma^2/2\right\} \approx -1'970'704.$$

If we write F_W for the distribution function of

$$W = \log Z \sim \mathcal{N}(\mu \approx 14.90425, \sigma^2 \approx 0.00011444),$$

we get using the translated log-normal approximation

$$F_S(x) = \mathbb{P}[S \le x] \approx \mathbb{P}[X \le x] = \mathbb{P}[k + Z \le x] = \mathbb{P}[\log Z \le \log(x-k)] = F_W(\log[x-k]),$$

for all $x > k$, and $F_S(x) = 0$ for all $x \leq k$. For all $\alpha \in (0,1)$ we then have

$$F_S^{-1}(\alpha) \approx k + \exp\left\{F_W^{-1}(\alpha)\right\}.$$

In particular, we get

$$q_{0.95} = F_S^{-1}(0.95) \approx k + \exp\left\{F_W^{-1}(0.95)\right\} \approx -1'970'704 + 3'023'266 = 1'052'562$$

and

$$q_{0.99} = F_S^{-1}(0.99) \approx k + \exp\left\{F_W^{-1}(0.99)\right\} \approx -1'970'704 + 3'045'387 = 1'074'684.$$

Note that since $k < 0$, the translated log-normal approximation in this example also allows for negative claims S, which under our model assumptions is excluded. The probability for negative claims S can be calculated as

$$F_S(0) \approx F_W(\log[0-k]) \approx F_W(\log 1'970'704) \approx 3.22 \cdot 10^{-322},$$

which is basically 0.

(c) We observe that with all the three approximations applied in Exercise 7.3 and in parts (a) and (b) above we get almost the same results. In particular, the normal approximation does not provide estimates that deviate significantly from the ones we get using the translated gamma and the translated log-normal approximations. This is due to the fact that $\lambda v = 1'000$ is large enough and the gamma distribution assumed for the claim sizes is not a heavy tailed distribution. Moreover, the skewness $\varsigma_S = 0.0321$ of S is rather small, hence the normal approximation is a valid model in this example. Note that in all the three approximations we allow for negative claims S, which actually should not be possible under our model assumptions. However, the probability to observe a negative claim S is vanishingly small in all the three approximations.