- EXERCISES -

SHEET 4

Exercise 1

Let $f : V \to \mathbb{R}$ bounded. Prove that

$$\forall n \geq 0 \quad \forall x \in V \quad E_x[f(X_{n+1})] = \frac{1}{d} \sum_{y \sim x} E_y[f(X_n)]$$

Exercise 2

- [Level 1] Prove that $\frac{1}{d} \leq \rho(G) \leq 1$
- [Level 2] Prove that $\rho(G) \geq \frac{1}{\sqrt{d}}$

Exercise 3

Define the operator $\mathrm{div} : \mathcal{X}_K \longrightarrow \mathcal{C}_K$ by

$$\forall \theta \in \mathcal{X}_K \quad \forall (x,y) \in \vec{E} \qquad (\mathrm{div}(\theta))(x) = \sum_{y \sim x} \theta((x,y))$$

1. Prove that div and ∇ are adjoint of each other

   ie. $\forall f \in \mathcal{C}_K \quad \forall \theta \in \mathcal{X}_K \quad \langle \theta, \nabla f \rangle_{\vec{E}} = \langle \mathrm{div}\,\theta, f \rangle$

2. Prove that $Id - P = \frac{1}{d}(\mathrm{div} \circ \nabla)$ and deduce

   $$\forall f, g \in \mathcal{C}_K \qquad \langle (Id - P) f, g \rangle = \frac{1}{d} \langle \nabla f, \nabla g \rangle_{\vec{E}}$$