

RANDOM WALKS
ON
TRANSITIVE GRAPHS -

Background:

- Probability theory. Applied stochastic processes (Markov chains).
- Group theory. (basic definitions \rightarrow group given by generators and relations.)

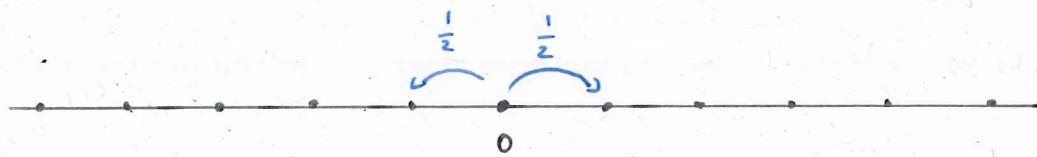
General goal

Understand the relationship between the geometric properties of a graph and the behavior of a simple random walk on this graph.

Framework

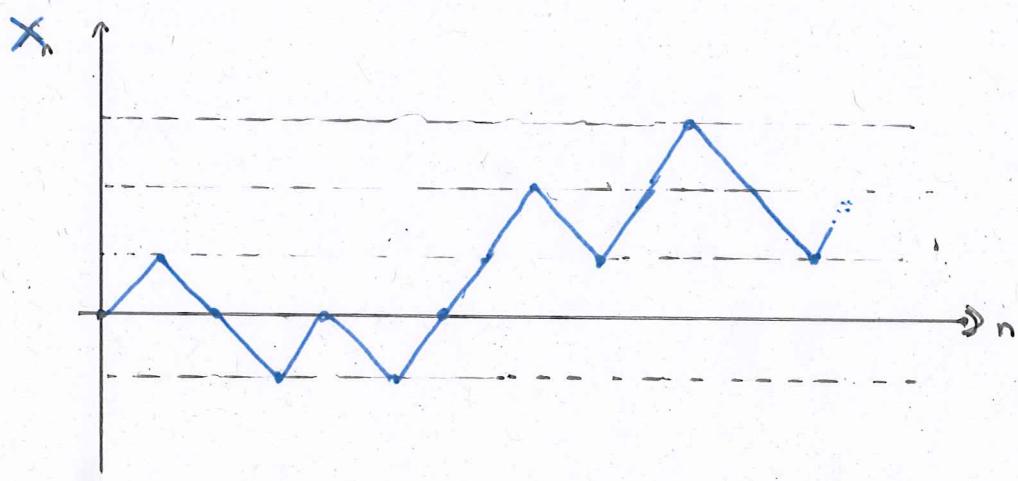
- $G = (V, E)$ infinite transitive graphs, $o \in V$ fixed origin.
 \rightarrow informal def.: "graph where all the vertices play the same role"
- (X_n) simple random walk on G , starting at o . (SRW)
 $\hookrightarrow X_0 = o$
 X_{n+1} = uniformly chosen neighbour of X_n .

Ex: SRW on \mathbb{Z}



Let $(z_i)_{i \geq 1}$ be iid r.v. with $P[z_i = -1] = P[z_i = +1] = \frac{1}{2}$.

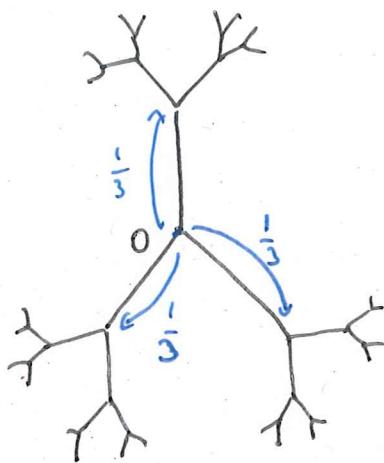
$X_n := \sum_{i=1}^n z_i$ is a SRW on \mathbb{Z} .



position of the walker after n steps.

Ex: SRW on \mathbb{Z}^d , $d \geq 1$.

Ex: SRW on the 3-regular tree T_3



QUESTION 11: RECURRENCE / TRANSIENCE ?

"Will the random walker visit the origin infinitely often?"

↳ on \mathbb{Z}, \mathbb{Z}^2 ; yes (the walk is recurrent)] [POLYA '21]

↳ on $\mathbb{Z}^3, \mathbb{Z}^4, \dots$; no (the walk is transient)

↳ on T^3 : no (transient)

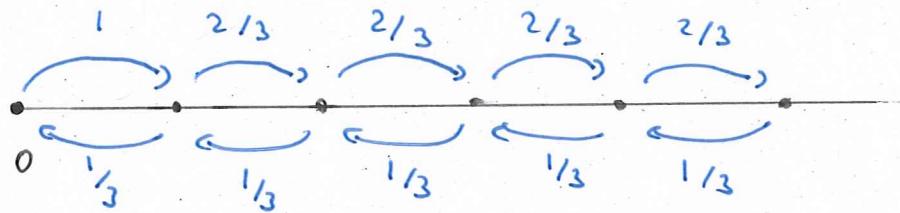
Idea of proof for the tree.

Let $(X_n)_{n \geq 0}$ be a SRW on \mathbb{T}^3 . Then

at each step (except when $X_n = 0$)

- jump away from 0 with proba $\frac{2}{3}$
- jump towards 0 with proba $\frac{1}{3}$

One can prove that the distance $|X_n|$ of the walker from 0 is a Markov Chain with transition probabilities



Then, using the Law of Large numbers, one can show that

$$\frac{|X_n|}{n} \xrightarrow{\text{a.s.}} \frac{1}{3}$$

and therefore, the walk is transient.

And for more general graphs?

Let $B_n = \{\text{vertices which can be reached by paths of length } \leq n\}$

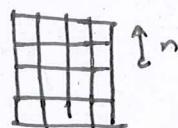
Ex on \mathbb{Z}

$$B_n = \dots \circ \circ \circ \circ \dots$$

$$|B_n| = 2n+1$$

\mathbb{Z}^d

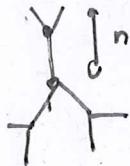
$$B_n =$$



$$|B_n| = (2n+1)^d$$

on \mathbb{T}^3

$$B_n =$$



$$\begin{aligned} |B_n| &= 1 + 3 + 6 + \dots + 3 \cdot 2^{n-1} \\ &= 3 \cdot 2^n - 2 \end{aligned}$$

We will prove that for general transitive graphs.

(the SRW on G is recurrent) $\iff (\exists C \text{ s.t. } \forall n |B_n| \leq Cn^2)$

"volume growth at most quadratic"

↑
property of the walk

↑
geometric property of
the graph.

QUESTION 2. WHAT IS THE RETURN PROBABILITY AFTER n STEPS?

$$p_n := P[X_n = 0] \quad \text{"return probability"}$$

Thm (admitted)

on \mathbb{Z}^d : $\exists c = c(d) > 0$. s.t.

$$p_n \underset{n \rightarrow \infty}{\sim} \frac{c(d)}{n^{1/2}} \quad \text{"poly. decay"}$$

on T^3 : we have

$$p_n \underset{n \rightarrow \infty}{\sim} \frac{1}{\sqrt{\pi n}} \left(\frac{8}{9}\right)^n \quad \text{"exponential decay"}$$

And (a more general graph)?

"edge boundary"

$$\text{Cheeger constant. } \phi := \inf_{S \subset V} \frac{|\partial S|}{|S|}$$

where the infimum is over all connected

subsets of V . and $\partial S = \{(x,y) \in E, x \in S, y \notin S\}$

Ex: for \mathbb{Z}^d $\phi = 0$ (take $S = B_n$)

for T^3 $\phi = 1$ (Hint: Use that for every S
 $\sum_{x \in S} \deg(x) = 2|E(S)| + |\partial S|$)

We will prove the following theorem, due to Kesten.

$$(\exists c > 0 \text{ s.t. } \forall n \ p_n \leq e^{-cn}) \Leftrightarrow (\underline{\Phi} > 0)$$

↑
"property of the SRW"

↑
"geometric graph property"

QUESTION 3 . SPEED OF THE SRW ?

$|X_n|$ = distance from 0 to X_n . What is

$$\rho := \lim_{n \rightarrow \infty} \frac{E[|X_n|]}{n} ?$$

(existence of the limit? Is $\rho > 0$?)

on \mathbb{Z} : The random walk can be written

$$X_n = z_1 + \cdots + z_n$$

where z_1, \dots, z_n iid uniform in $\{+1, -1\}$

Hence, the Law of Large numbers imply

$$\frac{X_n}{n} \xrightarrow[n \rightarrow \infty]{} E[z_i] = 0$$

and therefore, by dominated convergence

$$\frac{E[|X_n|]}{n} \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{"zero speed"}$$

Exercise: Prove that $\exists c > 0$ s.t.

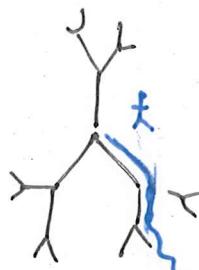
$$\lim_{n \rightarrow \infty} \frac{E[|X_n|]}{\sqrt{n}} = c \quad \text{"diffusive behaviour"}$$

on \mathbb{Z}^d , $d \geq 1$, equivalently.

$$\frac{E[X_n]}{n} \xrightarrow{n \rightarrow \infty} 0 \quad \text{"zero speed"}$$

on $\overline{\mathbb{T}}^3$. We have seen

$$\frac{E[X_n]}{n} \xrightarrow{n \rightarrow \infty} \frac{1}{3} \quad \text{"positive speed"}$$



"the random walker escapes quickly to infinity"

What happens on more general graphs?

A function $h : V \rightarrow \mathbb{R}$ is harmonic if

$$\forall x \in V \quad h(x) = \frac{1}{\deg(x)} \cdot \sum_{y \sim x} h(y)$$

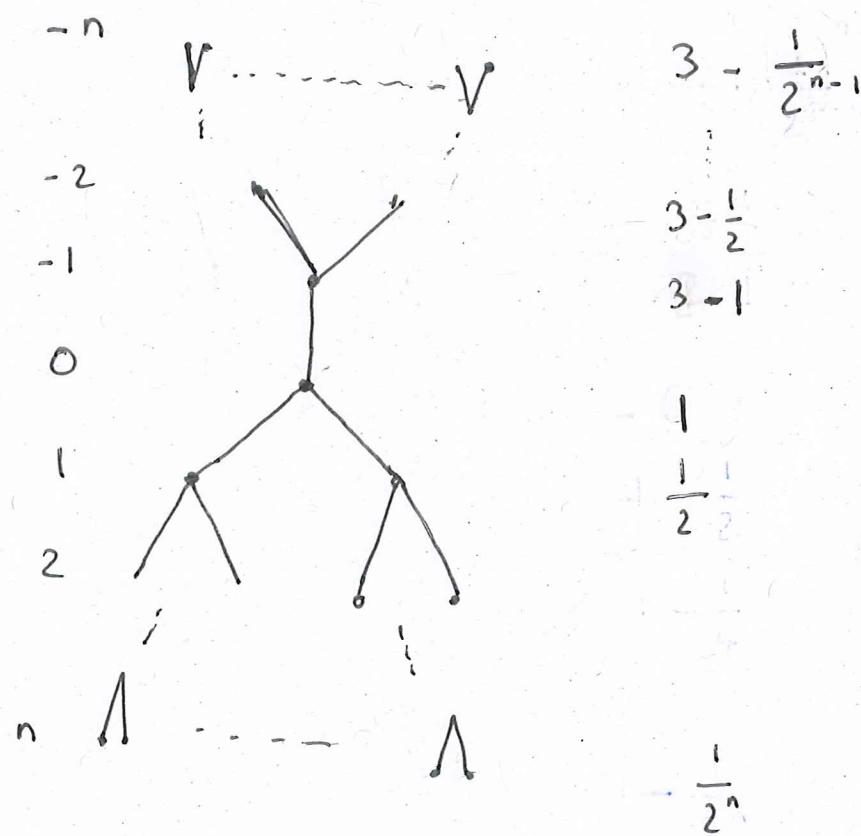
↑
neighborhood

Examples : $h = \text{cte}$ is always harmonic.

on \mathbb{Z} : $h(x) = \lambda x + \mu$

on \mathbb{Z}^2 : $h(x, y) = xy$

on \mathbb{T}^3



We will prove that

$$(f > 0)$$

"zero speed"

\iff (the only bounded harmonic functions are constant)

"Liouville property"

CHAPTER 1: TRANSITIVE GRAPHS

1) DEFINITIONS

Ref: [DIESTEL, Chap. 1] [GODSIL-ROYLE, Chap 1, 3]

Def: A graph is a pair $G = (V, E)$ where

- V is an arbitrary set
- E is a subset of $\{\{x, y\} : x, y \in V, x \neq y\}$

Notation. For $x, y \in V$ we write $xy := \{x, y\}$.

If $xy \in E$, we say that x and y are neighbours, and we write $x \sim y$.

Notice that the edges are unoriented ($xy = yx$)

Def: The degree of a vertex $x \in V$ is defined by

$$\deg(x) = |\{y \in V : y \sim x\}|.$$

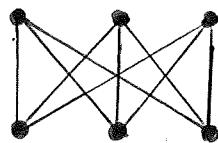
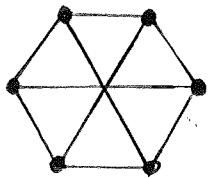
We say that G is locally finite if $\forall x \in V \deg(x) < \infty$.

Def: Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $\phi : V \rightarrow V'$ s.t.

$$(x \sim y \text{ in } G) \iff (\phi(x) \sim \phi(y) \text{ in } G')$$

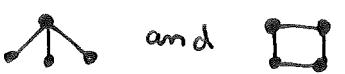
In this case, we say that ϕ is an isomorphism from G to G' .

Ex:



Diagrammatic representations of two isomorphic graphs

Rk: If ϕ isomorphism from G to G' , then $\deg_G(x) = \deg_{G'}(\phi(x))$

\hookrightarrow  and  are not isomorphic.

Def: An automorphism of $G = (V, E)$ is an isomorphism from G to itself.

Not: $\text{Aut}(G) = \{\text{automorphisms of } G\}$ group of automorphisms of G
(it is a subgroup of the group of the permutations of V)

Example:

$$G = \begin{array}{c} \text{graph} \\ \text{with } n \text{ vertices} \end{array}$$



$$\text{Aut}(G) = \{\tau^k, k=0 \dots n-1\} \cup \{\tau^k \circ \rho, k=0 \dots n-1\}$$

where $\tau(x) = x+1$ and $\rho(x) = -x$

"dihedral group of size $2n$ ".

Def: A graph $G = (V, E)$ is transitive iff

$$\forall x, y \in V \quad \exists \phi \in \text{Aut}(G) \text{ s.t. } \phi(x) = y$$

Rk: Equivalently, G is transitive if the action of $\text{Aut}(G)$ on V is transitive. (As a subgroup of the permutations of V , $\text{Aut}(G)$ acts naturally on V : for $\phi \in \text{Aut}(G)$, $x \in V$ $\phi \cdot x = \phi(x)$)

Rk: If G is transitive, $\deg(x)$ does not depend on $x \in V$ and is called the degree of G .

2. CAYLEY GRAPHS

Ref: [LYONS-PERES, Section 3.4] [SISTO] [DE LAHARIE, Chapters 2, 4]

Def: Let Γ be a group, let $S \subset \Gamma$

We say that S generates Γ if the smallest subgroup of Γ containing S is Γ .

S is symmetric if $S^{-1} = S$, i.e. $s \in S \Leftrightarrow s^{-1} \in S$

Def: A group Γ is finitely generated if $\exists S \subset \Gamma$ finite that generates Γ .

Def: Let Γ be a finitely generated group. Let $S \subset \Gamma$ be a finite symmetric set generating Γ .

The Cayley graph $\text{Cay}(\Gamma, S)$ associated to (Γ, S) is the graph with

- vertex set $V = \Gamma$
- edge set $E = \{(g, gs) : g \in \Gamma, s \in S\} = \{gg' : g, g' \in S\}$

Convention: we will always assume that the neutral element of Γ does not belong to S .

Rk: $\text{Cay}(\Gamma, S)$ is transitive (exercise)

- The definition above corresponds to the right Cayley graph (in the definition of the edges $\{g, gs\}$, the generator "s" on the "right") Alternatively, one could have defined the left-Cayley graph, which is isomorphic to the right Cayley graph, via $g \rightarrow g^{-1}$.

Examples

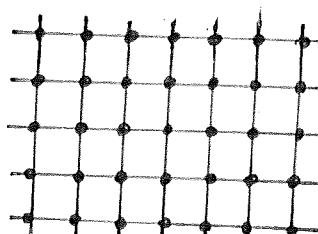
1) $\Gamma = \mathbb{Z}$ $S = \{-1, +1\}$ $\text{Cay}(\Gamma, S) = \dots \bullet \dots \bullet \dots \bullet \dots \bullet \dots$

2) $\Gamma = \mathbb{Z}$ $S = \{\pm 2, \pm 3\}$ $\text{Cay}(\Gamma, S) =$ 

"changing the generating set gives rise to a different Cayley graph!"

3) $\Gamma = \mathbb{Z}^2$ $S = \{\pm(0, 1), \pm(1, 0)\}$

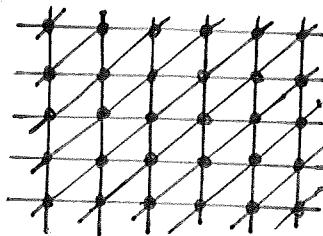
$\text{Cay}(\Gamma, S) =$



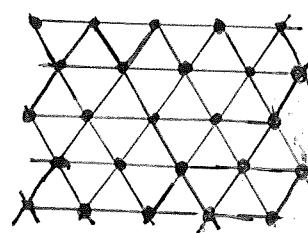
"square lattice"

4) $\Gamma = \mathbb{Z}^2$ $S = \{\pm(0, 1), \pm(1, 0), \pm(1, 1)\}$

$\text{Cay}(\Gamma, S) =$



\approx
isomorphic



"triangular lattice"

s) $\Gamma = \text{Aut}(\mathbb{Z})$ $S = \{t^{-1}, t, n\}$ where $t: \mathbb{Z} \rightarrow \mathbb{Z}$ $n: \mathbb{Z} \rightarrow \mathbb{Z}$

" $\Gamma \cong D_\infty$ the infinite dihedral group"

$\text{Cay}(\Gamma, S) =$

"ladder graph"

(Notice $nt^{-k} = t^k n$)

Rk: Two different groups may have the same Cayley graph (see e.g. D_∞ and $\mathbb{Z} \times \mathbb{Z}_{2\mathbb{Z}}$)

Free group over a finite set

Def: Let A be a finite set. Let $S = A \cup \{a^{-1}, a \in A\}$ (a^{-1} is just a formal symbol).

The set of words on S is the set $W(S)$ of finite sequences of elements of S . $W(S)$ is a monoid : the unit is the empty word, the product is given by the juxtaposition.

Let \sim be the equivalence relation on $W(S)$ generated by

$$ws s^{-1} w' \sim ww' \quad \text{and} \quad w s^{-1} s w' \sim ww'$$

for every $s \in A$ and every $w, w' \in W(S)$.

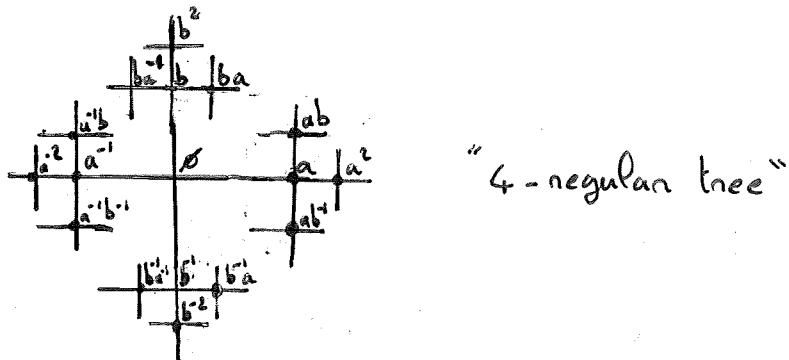
The Free group over A is defined by $\mathbb{F}_A = \frac{W(S)}{\sim}$

Exple: $A = \{a\}$ $S = \{a^{-1}, a\}$ $\mathbb{F}_A = \{a^k, k \in \mathbb{Z}\}$

$\text{Cay}(\mathbb{F}_A, S) =$

$A = \{a, b\}$ $S = \{a^{-1}, a, b^{-1}, b\}$ ($\mathbb{F}_A = \mathbb{F}_2$ free group on two elements)

$\text{Cay}(\mathbb{F}_2, S) =$



Groups defined by generators and relations

Let S be a finite set and $R \subseteq F_S$.

The group of presentation $\langle S | R \rangle$ is the quotient of F_S by the normal subgroup generated by R .

$$S = \{\text{"generators"}\} \quad R = \{\text{"relations"}\}$$

The presentation is said to be finite if S and R are finite.

Examples: $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ (^{not.} $\langle a, b \mid ab=ba \rangle$)

$$\mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$$
 (^{not.} $\langle a \mid a^n = 1 \rangle$)

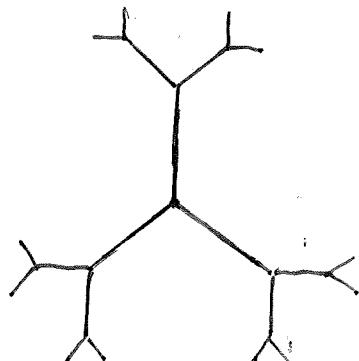
$$D_{2m} = \langle t, n \mid t^m, n^2, intnt^{-1} \rangle$$
 (^{not.} $\langle t, n \mid t^m = 1, n^2 = 1, ntnt^{-1} = t^{-1} \rangle$)

$$F_S = \langle S \mid \emptyset \rangle$$

Def. The free product of $\Gamma_1 = \langle S_1 \mid R_1 \rangle, \Gamma_2 = \langle S_2 \mid R_2 \rangle, \dots, \Gamma_k = \langle S_k \mid R_k \rangle$ is the group $\Gamma_1 * \Gamma_2 * \dots * \Gamma_k$ of presentation $\langle S_1 \cup S_2 \cup \dots \cup S_k \mid R_1 \cup R_2 \cup \dots \cup R_k \rangle$

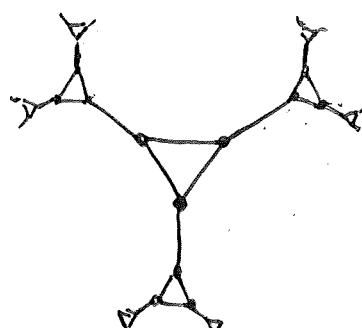
Rk: When a group is defined by a presentation $\langle S \mid R \rangle$ with S finite, it is finitely generated (by the image of S through the quotient). In particular we can define its Cayley graph.

Ex:



$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

$$(\langle a, b, c \mid a^2, b^2, c^2 \rangle)$$



$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$$

3) METRIC STRUCTURE

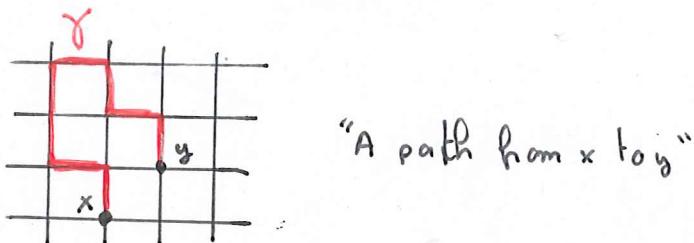
Ref [GODSIL - ROYLE, chap 1]

Let $G = (V, E)$ be a locally finite transitive graph.

Def: A path of length l from a vertex x to a vertex y is a sequence $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_l)$ of distinct vertices s.t.

$$\gamma_0 = x, \gamma_l = y \text{ and } \forall i: \gamma_{i-1} \sim \gamma_i.$$

Def: G is said to be connected if for every $x, y \in V$, there exists a path from x to y .



Def: The distance between two vertices x and y is defined by

$$d(x, y) = \min_{\gamma: x \rightarrow y} (\text{length}(\gamma))$$

where the minimum is over all the paths from x to y .

Rk: The distance d is invariant under the action of $\text{Aut}(G)$:

$$\forall \phi \in \text{Aut}(G) \quad \forall x, y \in V \quad d(\phi(x), \phi(y)) = d(x, y).$$

Pf: Let $\gamma = (\gamma_0, \dots, \gamma_l)$ be a path from x to y with $l = d(x, y)$. Then $\phi \cdot \gamma = (\phi \cdot \gamma_0, \dots, \phi \cdot \gamma_l)$ is a path from $\phi(x)$ to $\phi(y)$. Hence $d(\phi(x), \phi(y)) \leq d(x, y)$. The reverse inequality is obtained by considering ϕ^{-1} . □

4 GROWTH

Ref: [LYONS - PERES, p.472] [INRICH - STEIFER]

$G = (V, E)$ infinite, Locally-finite, transitive graph. $o \in V$ fixed origin.

Def. For $x \in V$, $m \geq 0$, the ball of radius m around x is defined by

$$B_m(x) = \{y \in V : d(x, y) \leq m\}$$

Not: $B_m = B_m(o)$.

Rk: By transitivity, the graphs induced by $B_n(x)$ and $B_n(y)$ are isomorphic (exercise). In particular $|B_n(x)| = |B_n(y)|$

Prop (Definition of the volume growth exponent)

The following limit exists and is finite:

$$v = \lim_{m \rightarrow \infty} \frac{1}{m} \log (|B_m|) \quad \text{"volume growth exponent"}$$

In other words $|B_m| = e^{vm + o(m)}$

Lem (Fekete's subadditivity Lemma.)

Let $(u_m)_{m \geq 0}$ be a sequence of numbers in $[-\infty, +\infty)$ satisfying

$$\forall m, n \geq 0 \quad u_{m+n} \leq u_m + u_n \quad \text{"subadditivity"}$$

Then the limit of $(\frac{u_m}{m})$ exists in $[-\infty, +\infty)$

$$\lim_{m \rightarrow \infty} \left(\frac{u_m}{m} \right) = \inf_{m \geq 0} \left(\frac{u_m}{m} \right)$$

Proof of the proposition:

We have $B_{m+n} = \bigcup_{x \in B_m} B_n(x)$. Hence $|B_{m+n}| \leq \sum_{x \in B_m} |B_n(x)| = |B_m| \cdot |B_n|$

Therefore $|B_{m+n}| \leq |B_m| \cdot |B_n|$. By applying Fekete's lemma

to $u_m = \log (|B_m|)$ we obtain $\frac{1}{m} \log (|B_m|) \rightarrow \inf_m \left(\frac{\log (|B_m|)}{m} \right)$
 $(u_m$ is finite because G is locally finite.)

Rk: $\forall n \ |B_n| \geq e^{vn}$.

Rk: We have $\forall n \geq 1 \ |B_{n+1}| \leq (d-1) |B_n| + 2$ where d is the degree of G .

Hence $v \leq \log(d-1)$ and the bound is realized for the d -regular tree. (the relation $|B_{n+1}| \leq (d-1) |B_n| + 2$ can be proved by induction)

Def: We say that G has

- exponential (volume) growth if $v > 0$
- polynomial (volume) growth if $\exists c < \infty$ s.t. $\forall n \ |B_n| \leq n^c$
- intermediate (volume) growth if $v = 0$ and $\forall c < \infty \ \sup_m \left(\frac{|B_m|}{m^c} \right) = +\infty$

Examples: • $T_d, d \geq 3$ has exponential growth

• $\mathbb{Z}^d, d \geq 1$ has polynomial growth.

• there exists Cayley graphs of intermediate growth
(Grigorchuk groups)

Thm: [Gromov, Traverso]

If G has polynomial volume growth, then $\exists k \in \mathbb{N} \ \exists c_1, c_2 > 0$ s.t.

$$\forall n \geq 0 \quad c_1 n^k \leq |B_n| \leq c_2 n^k.$$

- This deep theorem was proved by Gromov in the framework of groups, he showed that a Cayley has polynomial volume growth if and only if the underlying group is virtually nilpotent.
- It was later extended to transitive graphs by Traverso -

5 ISOPERIMETRIC CONSTANT.

Ref: [LEYBONS-PERES, Chapter 6] [PATERSON, Introduction]

$G = (V, E)$ infinite, locally-finite, transitive graph, $\deg(G) = d$.

Note: If $S \subset V$ finite, we write $\partial S = \{xy : x \in S, y \in V \setminus S\}$

Def: The isoperimetric constant of G is defined by

$$\Phi = \inf_{\substack{S \subset V \\ \text{finite}}} \frac{|\partial S|}{|S|}$$

We say that G is
 • amenable if $\Phi = 0$
 • non amenable if $\Phi > 0$.

Rk: There exist many characterisations of amenability, see [PATERSON].

The definition originates from group theory.

A group Γ is amenable if there exists an invariant mean on Γ , i.e. a positive linear map $\Lambda \in \text{Hom}(L^\infty(\Gamma), \mathbb{R})$ of norm 1 and satisfying $\forall x \in \Gamma \quad \forall f \in L^\infty(\Gamma) \quad \Lambda(x \cdot f) = \Lambda(f)$.

In the context of finitely generated groups the definitions coincide.

Prop: If G is non amenable, then it has exponential growth

$$\begin{aligned} |B_{m+1}| &= |B_m| + |\underbrace{\{x \in V : d(x, o) = m+1\}}_{\partial B_m}| \\ &\geq \frac{1}{d} \cdot |\partial B_m| \end{aligned}$$

$$\geq \left(1 + \frac{\Phi}{d}\right) |B_m|$$

$$\text{By induction } |B_n| \geq \left(1 + \frac{\Phi}{d}\right)^n \quad \square$$

6) THE LAMPLIGHTER GRAPH.

Ref: [LYONS-PERES p.88-89, p. 478] [WOESS 2005]

$G = (V, E)$ transitive locally-finite,

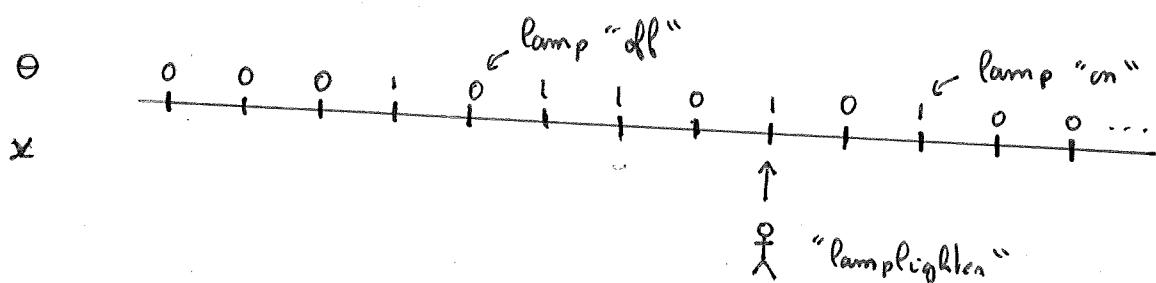
Not. Let \mathbb{H} be the set of functions $\Theta \in (\mathbb{Z}_{\geq 2})^V$ with finite support
(ie $\{x : \Theta_x \neq 0\}$ is finite)

Def: $LL(G)$ is the graph with vertex set $\mathbb{H} \times V$ and edge set defined by

$$(\Theta, x) \sim (\Theta', x') \iff \begin{cases} x = x' \text{ and } \Theta' = \Theta + \mathbf{1}_{\{x\}} \\ x \sim x' \text{ and } \Theta' = \Theta \end{cases}$$

Geometric interpretation (for $G = \mathbb{Z}$)

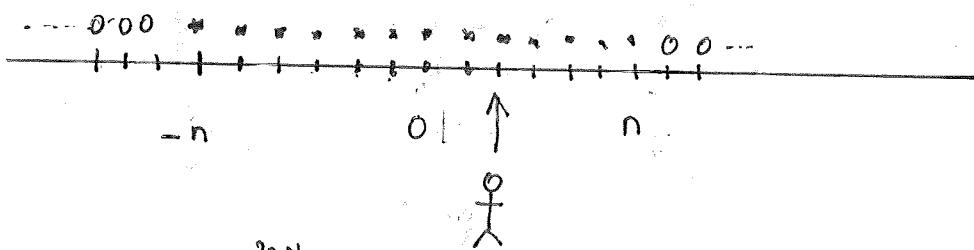
(Θ, x) vertex of $LL(G)$



- Possible moves:
- 1) the Lamplighter switches on or turns off the Lamp at its position.
 - 2) the Lamplighter moves to one of its neighbours.

Thm: (i) $LL(\mathbb{Z})$ has exponential volume growth.
(ii) $LL(\mathbb{Z})$ is amenable

Proof: Let $\Theta_n = \{\theta \in \Theta : \theta_x = 0 \text{ for } |x| > n\}$ and $S_n = \Theta_n \times \{-n, \dots, n\}$



We have $|S_n| = 2^{2n+1}$ and $S_n \subset B_{6n+1}$.

Hence $|B_{6n+1}| \geq 4^n$.

This concludes (c).

$$\text{Now } \partial S_n = \{(\theta, -n), (\theta, -n+1), \dots, (\theta, n), (\theta, n+1)\}, \theta \in \Theta_n \cup \{(\theta, n), (\theta, n+1)\}, \theta \in \Theta_n\}$$

$$|\partial S_n| = 2 \times 2^{2n+1}$$

$$\text{And therefore } \frac{|\partial S_n|}{|S_n|} = \frac{2}{2^{2n+1}} \xrightarrow{n \rightarrow \infty} 0 \text{ which proves (a)} \blacksquare$$

Rk1: The volume growth exponent of $LL(\mathbb{Z})$ can be computed exactly; Using that the set of vertices at distance exactly n from 0 "at the right of 0" is the Fibonacci sequence, one can show that $v = \log\left(\frac{1+\sqrt{5}}{2}\right)$, (see exercises)

Rk2: More generally if G is infinite $LL(G)$ has exponential volume growth and (G amenable) $\Leftrightarrow (LL(G)$ amenable)

Rk3 If (r, s) is a finitely generated group, there exists a group denoted $r \circ s \circ \frac{r}{s}$ and a natural generating "wedge product"

$$\text{set } \tilde{s} \text{ s.t. } \text{Cay}\left(r \circ s \circ \frac{r}{s}, \tilde{s}\right) = LL(\text{Cay}(r, s)).$$

"Lamplighter group": see [LYONS PERES p.88]

CHAPTER 2

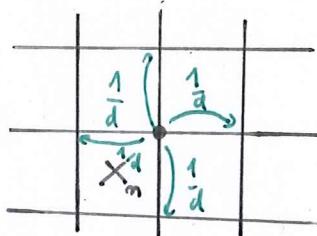
DEFINITIONS AND FIRST PROPERTIES

$G = (V, E)$ transitive, locally-finite, connected, infinite graph.
degree d , fixed origin $\omega \in V$.

1. DEFINITION

Def: Let $\omega \in V$. The simple random walk (SRW) on G starting at ω is the homogeneous Markov Chain $(X_n)_{n \geq 0}$ with

- state space V
- initial distribution S_ω
- transition probabilities $p(\omega, y) = \begin{cases} \frac{1}{d} & \text{if } \omega \sim y \\ 0 & \text{otherwise.} \end{cases}$



"At each step the walker jumps from its position to one of its neighbours, chosen uniformly and independently of the previous steps."

Rk: If $G = (V, E) = \text{Cay}(P, S)$ where S is a finite symmetric set generating the group P , then the random walk on G can be defined as follows. Let z_1, z_2, \dots be an iid sequence of uniform random variables on S . Then (X_n) , defined by

$$X_n = \omega + z_1 + z_2 + \dots + z_n,$$

is a simple random walk on G , starting at ω .

Convention: $X_n: \Omega \rightarrow V$ a.v. For every ω , we consider a proba. P_ω such that, under P_ω , $(X_n)_{n \geq 0}$ is a SRW starting at ω .

Rk: $\forall n \geq 0 \quad \omega_0, \dots, \omega_n \in V$

$$P_\omega[X_0 = \omega_0, \dots, X_n = \omega_n] = \prod_{i=0}^{n-1} p(\omega_i, \omega_{i+1}).$$

Note. For $m \geq 0$, $x, y \in V$, we write $\boxed{p_m(x, y) = P_x[X_m=y]}$
 and $\boxed{p_m(x) = p_m(x, x)}$

2. ELEMENTARY PROPERTIES

Ref: [ALDOUS-FILL]

2.1 SIMPLE MARKOV PROPERTY

Prop: Let $f: V^{\mathbb{N}} \rightarrow \mathbb{R}$ measurable bounded. For every $m \geq 0$, $x, y \in V$

$$\boxed{E_x[f((X_{m+n})_{n \geq 0}) | X_0, \dots, X_m] = E_y[f((X_n)_{n \geq 0}) \text{ on } \{X_m=y\}]}$$

P_x -a.s.

"Conditional on $\{X_m=y\}$, $(X_{m+n})_{n \geq 0}$ is a SRW starting at y , independent of X_0, \dots, X_{m-1} "

Consequence.

$\forall m, n \geq 0 \quad \forall x, y \in V$

$$\boxed{P_{m+n}(x, y) = \sum_{z \in V} p_m(x, z) p_n(z, y)} \quad [\text{Chapman-Kolmogorov}]$$

→ "matrix product interpretation"

Rk: A walk from x of length n is a sequence $\gamma = (\gamma_0, \dots, \gamma_n)$ such that $\gamma_0 = x$ and $\forall i \quad \gamma_{i-1} \sim \gamma_i$. (Contrary to a path, the vertices of a walk are not necessarily disjoint).

$$\begin{aligned} P_x[(X_0, \dots, X_n) = (x_0, \dots, x_n)] &= \prod_{x=x_0} \frac{1}{d^n} \prod_{x_0 \sim x_1} \dots \prod_{x_{n-1} \sim x_n} \\ &= \frac{1}{d^n} \prod_{\{(x_0, \dots, x_n) \text{ walk from } x \text{ of length } n\}} \end{aligned}$$

" (x_0, \dots, x_n) is a uniformly chosen walk from x of length n "

2.2 IRREDUCIBILITY

Prop: The SRW is an irreducible Markov Chain:

$$\forall x, y \in V \quad \exists m \geq 0 : p_m(x, y) > 0$$

Proof: Since $G = (V, E)$ is connected, there exists a path $\gamma = (\gamma_0, \dots, \gamma_m)$ from x to y .

$$p_m(x, y) = P_x [X_m = y]$$

$$\geq P_x [X_0 = \gamma_0, \dots, X_m = \gamma_m]$$

$$= \frac{1}{d^n} > 0$$

■

2.3 APERIODICITY

Recall that the period of (X_n) is defined as

$$\gcd \{m \geq 0 : p_m(0) > 0\}$$

Since $\forall k \geq 0 \quad p_{2k}(0) > 0$, the period of the SRW is 1 (aperiodic case) or 2. The following proposition gives geometric conditions on G characterizing the aperiodic case.

Def: A cycle is a path $\gamma = (\gamma_0, \dots, \gamma_m)$ s.t. $\gamma_0 = \gamma_m$

Prop: The following are equivalent.

- (i) the SRW is aperiodic
- (ii) there exists an odd cycle in G
- (iii) G is not bipartite

If (i)-(ii) do not hold, then the SRW has period 2.

Proof: (i) \Rightarrow (ii)

If the SRW is aperiodic, there exists m odd such that

$$P_0(X_m = 0) > 0, \text{ Hence } \exists (x_0, \dots, x_m) \text{ s.t. } x_0 = x_m = 0$$

$$\text{and } P_0(X_0 = x_0, X_1 = x_1, \dots, X_m = x_m) > 0$$

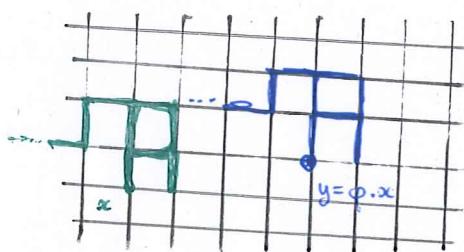
Therefore (x_0, \dots, x_m) is an odd walk satisfying $x_0 = x_m$, and G contains an odd cycle (exercise).

2.4 INVARIANCE

Propn: $\forall \varphi \in \text{Aut}(G) \quad p_m(\varphi \cdot x, \varphi \cdot y) = p_m(x, y)$

Consequence: $\forall m \quad p_m(\varphi \cdot x, \varphi \cdot y) = p_m(x, y)$

- Let $\varphi \in \text{Aut}(G)$ be such that $\varphi \cdot x = y$. If $(X_n)_{n \geq 0}$ is a SRW starting at x , then $(\varphi \cdot X_n)_{n \geq 0}$ is a SRW starting at y .



"in \mathbb{Z}^2 , the translate of a SRW from x is a SRW from y "

2.5 REVERSIBILITY

Propn: $\forall x, y \in V \quad p(x, y) = p(y, x)$

"the measure μ on V defined by $\mu_x = 1$ $\forall x \in V$ is reversible for the SRW"

We say that the SRW is reversible (see [Aldous-Fill, chapter 3])

Consequences / geometric interpretation.

$$\forall m \geq 0 \quad \forall x, y \in V \quad p_m(x, y) = p_m(y, x)$$

$$\forall (x_0, x_1, \dots, x_m) \in V^{m+1}$$

$$P_x [(x_0, \dots, x_m) = (x_0, \dots, x_m)] = P_y [(x_0, \dots, x_m) = (x_m, \dots, x_0)]$$



"the law of a m -step walk from x to y is that of an m -step walk from y to x ."

Application:

$$\forall m \geq 0 \quad \forall x \in V \text{ we have } p_{2m}(0,0) \geq p_{2m}(0,x)$$

Proof:

$$p_{2m}(0,0) = \sum_{y \in V} p_m(0,y) p_m(y,0) \quad (\text{Chapman - Kolmogorov})$$

$$= \sum_{y \in V} p_m(0,y)^2 \quad (\text{Reversibility})$$

$$= \sqrt{\sum_{y \in V} p_m(0,y)^2} \cdot \sqrt{\sum_{y \in V} p_m(x,y)^2} \quad (\text{Invariance} \\ + \text{transitivity of } G)$$

$$\geq \sum_{y \in V} p_m(0,y) p_m(x,y) \quad (\text{Cauchy - Schwarz})$$

$$= p_{2m}(0,x) \quad (\text{Chapman - Kolmogorov} + \text{Reversibility}) \blacksquare$$

2.6 STRONG MARKOV PROPERTY.

Let $(X_n)_{n \geq 0}$ be a SRW starting at x , $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$, $\mathcal{F}_\infty = \sigma(X_n)_{n \in \mathbb{N}}$.

A stopping time T is a random variable $T \in \mathbb{N}$ o.t. $\{T=m\} \in \mathcal{F}_m$.

Define $\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T=m\} \in \mathcal{F}_m \text{ for each } m \geq 0\}$.

Prop: Let $f : V^{\mathbb{N}} \rightarrow \mathbb{R}$ measurable bounded. For every $x, y \in V$,

$$E_x [f((X_{T+m})_{m \geq 0})] | \mathcal{F}_T = E_y [f((X_m)_{m \geq 0})] \text{ on } \{T < \infty, X_T = y\}$$

"Conditioned on $T < \infty$ and $X_T = y$, X_{T+m} is a SRW starting at y , independent of \mathcal{F}_T "

2.7. RECURRENCE / TRANSIENCE

(17)

$$H_0^+ = \min \{ n \geq 1 : X_n = 0 \}$$

Def. We say that the SRW on G is

- recurrent if $P_0 [H_0^+ < \infty] = 1$,
- transient if $P_0 [H_0^+ < \infty] < 1$.

$$V_0 := \sum_{n \geq 0} \mathbb{1}_{X_n=0}, \quad P_0 = P_0 [H_0^+ < \infty]$$

By the strong Markov property $P_0 [V_0 \geq n] = P_0^{n-1}, \quad n \geq 1$

$$\rightarrow E_0 [V_0] = \sum_{n=1}^{\infty} P_0 [V_0 \geq n] = \frac{1}{1 - P_0}$$

$$\rightarrow E_0 [V_0] = E_0 \left[\sum_{n \geq 0} \mathbb{1}_{X_n=0} \right] = \sum_{n \geq 0} P_0 [X_n = 0] = \sum_{n \geq 0} p_n(0)$$

Thm: The following are equivalent.

- (i) the SRW is transient.
 - (ii) $\sum_{n=0}^{\infty} p_n(0) < \infty$.

CHAPTER 3:

RETURN PROBABILITY.

Ref: [LYONS-PERES]

$G = (V, E)$ transitive, loc.-finite, connected, infinite graph
degree d (except when $G = \mathbb{Z}^d$), fixed origin $\sigma \in V$.

The goal of this chapter is to prove the following theorem:

Thm [Kesten '58]

The following are equivalent:

- (i) $\exists c > 0$ s.t. $\forall n \geq 0 \quad P_\sigma[X_n = 0] \leq e^{-cn}$;
- (ii) G is non-amenable.

1) SPECTRAL RADIUS.

1.1) Transition operator.

Motivation: If G finite graph

transition matrix: $(P_{xy})_{x,y \in V} = (p(x,y))_{x,y \in V}$



associated
linear operator

$$\begin{aligned} P : \mathbb{R}^V &\longrightarrow \mathbb{R}^V \\ (f_x)_{x \in V} &\mapsto \underbrace{\left(\sum_{y \in V} p(x,y) f_y \right)}_{E_x[f(x)]}_{x \in V} \end{aligned}$$

Note: $\mathcal{C}_k = \{f: V \rightarrow \mathbb{R} \text{ with finite support}\}$

$$\text{For } f, g \in \mathcal{C}_k \quad \langle f, g \rangle = \sum_{x \in V} f(x)g(x). \quad \|f\|_2 = \sqrt{\langle f, f \rangle}$$

Def: The transition operator $P: \mathcal{C}_k \rightarrow \mathcal{C}_k$ associated to the SRW on G is defined by

$$\forall f \in \mathcal{C}_k \quad \forall x \in V \quad (Pf)(x) = E_x [f(X_1)]$$

Rk: equivalently $(Pf)(x) = \sum_{y \in V} p(x, y) f(y) = \frac{1}{d} \sum_{y \sim x} f(y).$

Probabilistic interpretation.

If $\pi = (\pi(x))_{x \in V}$ is a finitely supported probability measure on V .

If x_0 is sample according to π then $P\pi$ is the law of X_1 obtained from x_0 by doing one step of the SRW.

In particular for $x, y \in V$

$$p_m(x, y) = \langle P^m \mathbf{1}_{\{x\}}, \mathbf{1}_{\{y\}} \rangle \quad (*)$$

Note: $\|P\|_2 = \sup_{f \in \mathcal{C}_k \setminus \{0\}} \left\{ \frac{\|Pf\|_2}{\|f\|_2} \right\}$ " ℓ^2 -operator norm"

Rk: The operator P can be defined on $\ell^2(V)$.

Properties: (i) P is a self-adjoint operator on \mathcal{C}_k :

$$\forall f, g \in \mathcal{C}_k \quad \langle f, Pg \rangle = \langle Pf, g \rangle$$

$$(ii) \quad \|P\|_2 = \sup_{f \in \mathcal{C}_k \setminus \{0\}} \left\{ \frac{|\langle Pf, f \rangle|}{\langle f, f \rangle} \right\} = \sup_{f \in \mathcal{C}_k \setminus \{0\}} \left\{ \frac{\langle Pf, f \rangle}{\langle f, f \rangle} \right\}$$

"Rayleigh Quotient"

Proof (ii)

(20)

$$\text{Let } C = \sup_{f \in E_2 \setminus \{0\}} \frac{|\langle Pf, f \rangle|}{\langle f, f \rangle}$$

By Cauchy-Schwarz, $\forall f \neq 0 \quad |\langle Pf, f \rangle| \leq \|Pf\|_2 \|f\|_2^2$

$$\text{Hence } C \leq \|P\|_2$$

$$\text{Now Set } g = \frac{\|f\|}{\|Pf\|} \cdot Pf$$

$$\|Pf\| \|f\| = \langle Pf, g \rangle$$

$$= \frac{1}{4} (\langle P(f+g), f+g \rangle - \langle P(f-g), f-g \rangle)$$

$$\leq C + \frac{1}{4} (\|f+g\|^2 + \|f-g\|^2)$$

$$\leq C \|f\|^2$$

$$\text{Hence } \|P\|_2 \leq C$$

$$\begin{aligned} \text{Finally observe that } \langle Pf, f \rangle &= \left| \sum_x (Pf)(x) f(x) \right| \\ &= \left| \sum_x E_x [f(x)] f(x) \right| \\ &\leq \sum_x |E_x [f(x)]| |f(x)| \\ &= \langle f, |f| \rangle \end{aligned}$$

$$\text{And therefore } C = \sup_{f \neq 0} \frac{\langle Pf, f \rangle}{\|Pf\|^2}$$

■

$$\begin{aligned}
 \text{Proof: (i)} \quad & \langle f, p_g \rangle = \sum_{x \in V} f(x) (p_g)(x) \\
 &= \sum_{x,y \in V} p(x,y) f(x) g(y) \\
 &= \sum_{x,y \in V} p(y,x) f(x) g(y) \quad (\text{Reversibility}) \\
 &= \sum_{y \in V} (p f)(y) g(y) \\
 &= \langle p f, g \rangle
 \end{aligned}$$

□

1.2 SPECTRAL RADIUS

Def. The spectral radius of the SRW on G is defined by

$$\rho(G) = \limsup_{n \rightarrow \infty} (p_n(0))^{\frac{1}{n}}$$

Rk.: In particular $p_n(0) \leq \rho(G)^{m+o(n)}$

("exponential decay of the return probability") $\Leftrightarrow (\rho(G) < 1)$

Rk.: By irreducibility, we have $\forall x, y \in V$

$$\rho(G) = \limsup_{n \rightarrow \infty} p_m(x,y)^{\frac{1}{n}}$$

Rk: The "limsup" is important if the graph G is bipartite
 $(p_{2m+1}(0) = 0)$

Exercise: Prove that $\rho(G) = \lim_{n \rightarrow \infty} (p_{2n}(0))^{\frac{1}{2n}}$

and if G is not bipartite $\rho(G) = \lim_{n \rightarrow \infty} (p_n(0))^{\frac{1}{n}}$.

• Prove that $\frac{1}{d} \leq \rho(G) \leq 1$.

Theorem:

$$\rho(G) = \|P\|_2 \text{ and } \forall m \ \forall x, y \quad p_m(x, y) \leq \|P\|_2^m$$

We will need the following elementary lemma.

Lemma:

(1) If (u_n) is an increasing sequence of real numbers

then $\ell = \lim_{n \rightarrow \infty} u_n$ exists and $\ell = \lim_{m \rightarrow \infty} (\prod_{k=m}^n u_k)^{\frac{1}{n-m}}$. (cesaro)

(2) Let $(u_n^{(1)}), \dots, (u_n^{(k)})$ be sequences of nonnegative real numbers

$$\forall a_1, \dots, a_k > 0 \quad \limsup_{n \rightarrow \infty} (a_1 u_n^{(1)} + \dots + a_k u_n^{(k)})^{\frac{1}{n}} = \max_k (\limsup_{n \rightarrow \infty} (u_n^{(k)})^{\frac{1}{n}})$$

$$\text{Proof: } p_m(x, y) = \langle P^m \mathbf{1}_x, \mathbf{1}_y \rangle$$

$$\leq \|P^m \mathbf{1}_x\|_2 \| \mathbf{1}_y \|_2. \quad (\text{By Cauchy-Schwarz})$$

$$\leq \|P\|_2^m.$$

This implies $\underline{\rho(G) \leq \|P\|_2}$ and the second part of the theorem.

It remains to prove $\|P\|_2 \leq \rho(G)$.

Let $f \in \mathcal{B}_k \setminus \{0\}$, $f \geq 0$

$$\begin{aligned} \|P^{m+1} f\|_2^2 &= \langle P^{m+1} f, P^{m+1} f \rangle \\ &= \langle P^m f, P^{m+2} f \rangle \\ &\stackrel{\text{CS}}{\leq} \|P^m f\|_2 \|P^{m+2} f\|_2 \end{aligned}$$

$$\text{Hence } \frac{\|P^{m+1} f\|_2}{\|P^m f\|_2} \leq \frac{\|P^{m+2} f\|_2}{\|P^{m+1} f\|_2}.$$

Applying (1) in the lemma to $u_n = \frac{\|P^n f\|_2}{\|P^{n-1} f\|_2}$

$$\text{we have } u_1 \leq \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{\|P^n f\|_2}{\|P^{n-1} f\|_2} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|P^n f\|_2^{\frac{1}{n}}.$$

$$\begin{aligned}
 \text{Now } \|P^n f\|_2^{\frac{1}{2^n}} &= \langle P^n f, P^n f \rangle^{\frac{1}{2^n}} \\
 &= \langle P^{2^n} f, f \rangle^{\frac{1}{2^n}} \\
 &= \left(\sum_{x,y \in V} f(x) f(y) p_{2^n}(x,y) \right)^{\frac{1}{2^n}} \leq \left(\sum_{x,y \in V} |f(x)|^2 |f(y)|^2 p_{2^n}(x,y) \right)^{\frac{1}{2^n}}
 \end{aligned}$$

By applying (2) in Lemma, we obtain.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \|P^n f\|_2^{\frac{1}{2^n}} &\leq \max_{x,y} \left(\limsup_{n \rightarrow \infty} (p_{2^n}(x,y))^{\frac{1}{2^n}} \right) \\
 &= \max_x \limsup_{n \rightarrow \infty} p_{2^n}(x)^{\frac{1}{2^n}} \\
 &= \limsup_{n \rightarrow \infty} p_{2^n}(0)^{\frac{1}{2^n}} \quad (\text{because } p_{2^n}(x) \leq p_{2^n}(0)) \\
 &= \rho(G)
 \end{aligned}$$

Therefore $\frac{\|Pf\|_2}{\|f\|_2} \leq \rho(G)$ which proves $\underline{\|P\|_2 \leq \rho(G)}$

□

13 Gradient operator.

Not: We consider the set of oriented edges:

$$\boxed{\vec{E} = \{(x,y) : xy \in E\}.}$$

If $e = (x,y) \in \vec{E}$, we write $-e = (y,x)$.

(a non-oriented edge $xy \in E$ is associated to two oriented edges $e = (x,y)$ and $-e = (y,x)$)

Notation: Let \mathcal{C}_K be the set of functions $\Theta: \vec{E} \rightarrow \mathbb{R}$ s.t.

- $\forall e \in \vec{E} \quad \Theta(-e) = -\Theta(e)$, [antisymmetry]

- $\{\epsilon \in \vec{E}: \Theta(\epsilon) \neq 0\}$ is finite. [Finite support]

- $\forall \Theta, \Psi \in \mathcal{C}_K \quad \langle \Theta, \Psi \rangle_{\vec{E}} = \frac{1}{2} \sum_{e \in \vec{E}} \Theta(e) \Psi(e).$

- $\forall \Theta \in \mathcal{C}_K \quad \|\Theta\|_2 = \sqrt{\langle \Theta, \Theta \rangle_{\vec{E}}}.$

Def: [Gradient operator]

We define the operator $\nabla: \mathcal{C}_K \longrightarrow \mathcal{C}_K$ by

$$\forall f \in \mathcal{C}_K \quad \forall (x, y) \in \vec{E} \quad (\nabla f)(x, y) = f(y) - f(x).$$

Lemma [relation between P and ∇]

For every $f, g \in \mathcal{C}_K$ we have

$$\langle (Id - P)f, g \rangle = \frac{1}{d} \langle \nabla f, \nabla g \rangle_{\vec{E}}.$$

Rk: $Id - P \stackrel{\text{def}}{=} \Delta$ "discrete Laplacian"

The equation above can be seen as a discrete integration by parts.

Proof: For $x \in V$ $[(Id - P)f](x) = f(x) - \sum_{y \in V} p(x, y) f(y)$

$$= \sum_{y \in V} p(x, y) [f(x) - f(y)].$$

$$\text{Hence } \langle (\text{Id} - P) f, g \rangle = \sum_{x, y \in V} p(x, y) [f(x) - f(y)] g(x)$$

$$= \frac{1}{2} \sum_{x, y \in V} p(x, y) [f(x) - f(y)] g(x)$$

$$+ \frac{1}{2} \sum_{x, y \in V} p(y, x) \underbrace{[f(y) - f(x)]}_{= p(x, y)} g(y)$$

$$= \frac{1}{2} \sum_{x, y \in V} p(x, y) \underbrace{[f(x) - f(y)]}_{= \frac{1}{d} \|\nabla f\|_2^2} [g(x) - g(y)]$$

$$= \frac{1}{d} \langle \nabla f, \nabla g \rangle_E.$$

Proposition: [Variational formula for $\rho(G)$]

We have

$$\rho(G) = 1 - \frac{1}{d} \left(\inf_{f \in \mathcal{P}_k \setminus \{0\}} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \right)^2.$$

Proof:

Using Rayleigh quotient we find

$$\rho(G) = \|P\|_2 = \sup_{f \in \mathcal{P}_k \setminus \{0\}} \left\{ \frac{\langle Pf, f \rangle}{\langle f, f \rangle} \right\}$$

$$= 1 - \inf_{f \in \mathcal{P}_k \setminus \{0\}} \left\{ \frac{\langle (\text{Id} - P)f, f \rangle}{\langle f, f \rangle} \right\}$$

Lemma

$$= 1 - \inf_{f \in \mathcal{P}_k \setminus \{0\}} \left\{ \frac{1}{d} \frac{\langle \nabla f, \nabla f \rangle_E}{\langle f, f \rangle} \right\}$$

2 ISOOPERIMETRIC CONSTANT REVISITED

Note. For $\Theta \in \mathcal{E}_k$ write

$$\|\Theta\|_1 = \frac{1}{2} \sum_{e \in \vec{E}} |\Theta(e)|,$$

and for $f \in \mathcal{C}_k$ write

$$\|f\|_1 = \sum_{x \in V} |f(x)|.$$

$$\text{Prop: } \Phi = \inf_{f \in \mathcal{C}_k \setminus \{0\}} \frac{\|\nabla f\|_1}{\|f\|_1}$$

Proof: $\boxed{\geq}$ Let $S \subset V$ be finite non empty, then $1_S \in \mathcal{C}_k \setminus \{0\}$ and

$$\cdot \|\nabla 1_S\|_1 = |\partial S|$$

$$\cdot \|1_S\|_1 = |S|$$

$$\text{Hence } \Phi = \inf_{\substack{S \subset V \\ \text{finite}}} \frac{\|\nabla 1_S\|_1}{\|1_S\|_1} \geq \inf_{f \in \mathcal{C}_k \setminus \{0\}} \frac{\|\nabla f\|_1}{\|f\|_1}.$$

$\boxed{\leq}$ Let $f \in \mathcal{C}_k \setminus \{0\}$.

For $t > 0$, consider $S_t = \{x \in V : |f(x)| > t\}$

S_t is finite (because $f \in \mathcal{C}_k$) and by definition

$$\phi \cdot |S_t| \leq |\partial S_t|,$$

which can be rewritten as

$$\phi \cdot \sum_{x \in V} \mathbb{1}_{\{|f(x)| > t\}} \leq \sum_{(x,y) \in \vec{E}} \mathbb{1}_{\{f(x) > t \geq f(y)\}}$$

Integrating from $t=0$ to $t=+\infty$ and using Fubini theorem,
we obtain

$$\begin{aligned}
 \phi \cdot \underbrace{\sum_{x \in V} \int_0^\infty \mathbb{1}_{\{f(x) > t\}} dt}_{= \|f\|_1} &\leq \underbrace{\sum_{(x,y) \in \vec{E}} \int_0^\infty \mathbb{1}_{\{f(x) > t \geq f(y)\}} dt}_{= \sum_{(x,y) \in \vec{E}} (|f(x)| - |f(y)|) \mathbb{1}_{\{f(x) > f(y)\}}} \\
 &= \frac{1}{2} \sum_{(x,y) \in \vec{E}} \left| |f(x)| - |f(y)| \right| \\
 &\leq \|\nabla f\|_1.
 \end{aligned}$$

Therefore, $\phi \leq \inf_{f \in \mathcal{C}_k \setminus \{0\}} \frac{\|\nabla f\|_1}{\|f\|_1}$.

3 PROOF OF KESTEN'S THEOREM.

We prove the following stronger theorem.

Thm:

$$1 - \left(\frac{\phi}{d}\right) \stackrel{(i)}{\leq} \rho(G) \stackrel{(ii)}{\leq} \sqrt{1 - \left(\frac{\phi}{d}\right)^2}$$

$$\text{Proof: (i)} \quad \rho(G) = 1 - \frac{1}{d} \inf_{f \in \mathcal{C}_k \setminus \{0\}} \left\{ \frac{\langle \nabla f, \nabla f \rangle_{\vec{E}}}{\langle f, f \rangle} \right\}$$

$$\geq 1 - \frac{1}{d} \inf_{\substack{S \subset V \\ \text{finite}}} \left\{ \frac{\langle \nabla \mathbb{1}_S, \nabla \mathbb{1}_S \rangle_{\vec{E}}}{\langle \mathbb{1}_S, \mathbb{1}_S \rangle} \right\} = 1 - \frac{\phi}{d}.$$

(2) Let $f \in \mathcal{C}_k \setminus \{0\}$.

$$\|f\|_2^2 = \|f^2\|_1 \quad \Phi \leq \|\nabla f^2\|_1$$

$$= \frac{1}{2} \sum_{(x,y) \in \tilde{E}} |f(x) - f(y)| |f(x) + f(y)|$$

$$\stackrel{\text{CS}}{\leq} \|\nabla f\|_2^2 \cdot \sqrt{\frac{1}{2} \sum_{(x,y) \in \tilde{E}} f(x)^2 + f(y)^2 + 2f(x)f(y)}$$

$$= \|\nabla f\|_2^2 \cdot \sqrt{d \sum_{x \in V} f(x)^2 + \underbrace{\sum_{(x,y) \in \tilde{E}} f(x)f(y)}_{= d \cdot \sum_{x,y \in V} p(x,y) f(x)f(y)}}$$

$$= d \cdot \langle Pf, f \rangle$$

$$= \|\nabla f\|_2^2 \cdot \sqrt{2d \|f\|_2^2 - \|\nabla f\|_2^2}$$

Dividing the expression above by $\|f\|_2^2$ and setting $\alpha = \frac{\|\nabla f\|_2^2}{\|f\|_2^2}$,

$$\text{we obtain } \phi \leq \alpha \cdot \sqrt{2d - \alpha^2},$$

$$\text{i.e. } d^2 - \phi^2 \geq (d - \alpha^2)^2.$$

Dividing by d^2 , we finally get $1 - \left(\frac{\phi}{d}\right)^2 \geq \left(1 - \frac{1}{d} \frac{\|\nabla f\|_2^2}{\|f\|_2^2}\right)^2$.

Taking the sup over $f \in \mathcal{C}_k \setminus \{0\}$ concludes that

$$1 - \left(\frac{\phi}{d}\right)^2 \geq \rho(G)^2$$

CHAPTER 4 :

THE VAROPOULOS - CARNE BOUND.

Ref: [LYONS-PERES] [Blog, T. TAO]

$G = (V, E)$ transitive, locally-finite, connected, infinite graph
degree d , fixed origin $o \in V$.

$(X_m)_{m \geq 0}$ SRW on G , P transition operator, $p_m(x, y) = P_x[X_m=y]$.

Motivation: In the previous chapters, we have seen

$$p_m(x, y) \leq p_m(o, o) \leq \|P\|_2^m$$

We expect $p_m(x, y)$ to be "substantially" smaller than $\|P\|_2^m$ when x and y are far from each other.

(In particular, we trivially have $p_m(x, y) = 0$ if $d(x, y) > m$)

In this chapter, we prove the following inequality, which improves quantitatively the bound $p_m(x, y) \leq \|P\|_2^m$ when $d(x, y) \geq \sqrt{2m}$

It is a generalization and improvement by Carne (1985) of a result of Varopoulos (1985). The statement below is a refinement of Lyons and Peres (see Thm 13.4 in [LYONS-PERES]).

Thm: For all $x, y \in V$, for all $m \geq 0$, we have:

$$p_m(x, y) \leq 2 \|P\|_2^m \exp\left(-\frac{d(x, y)^2}{2m}\right)$$

[Varopoulos - Carne Bound]

We will see several applications in the next chapters:

. When G has polynomial growth, it implies that $P_o[d(o, X_m) \leq 4\sqrt{m \log_2}] \rightarrow 1$

. When G has sub-exponential growth, it implies that X_m has speed $\ell=0$.

1. The case $G = \mathbb{Z}$.

Thm: Let $(x_n)_{n \geq 0}$ be a SRW on \mathbb{Z} , starting at 0. Then

$$P_0 [|x_n| \geq d] \leq 2 e^{-\frac{d^2}{2n}}$$

Proof: Let z_1, z_2, \dots, z_m be iid random variables s.t. $P[z_i = 1] = P[z_i = -1] = \frac{1}{2}$.

Let $s = \frac{d}{n}$. By symmetry

$$P_0 [|x_n| \geq d] = 2P [s z_1 + \dots + s z_m \geq s^2 n]$$

$$= 2P [e^{s z_1 + \dots + s z_m} \geq e^{s^2 n}]$$

$$\leq 2 \left(\frac{E[e^{s z_1}]}{e^{s^2}} \right)^n \quad [\text{By Markov inequality + independence}]$$

Using that $E[e^{s z_1}] = \cosh(s) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} s^{2k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k k!} s^{2k} = e^{s^2/2}$,

we finally get $P_0 [|x_n| \geq d] \leq 2 e^{-ns^2/2}$

2. Chebyshev polynomials.

Definition (Chebyshev Polynomials)

We consider the sequence of polynomials $(T_n)_{n \in \mathbb{Z}}$ defined by

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+2}(x) = 2x T_{n+1}(x) - T_{n-1}(x) \end{cases} \quad \text{and } T_{-n} = T_n$$

Property: For every $\theta \in \mathbb{R}$

$$\cos(n\theta) = T_n(\cos \theta)$$

Proof: By induction (use $\cos((m+1)\theta) + \cos((m-1)\theta) = 2 \cos \theta \cos m\theta$)

Properties:

(i) $T_m > 0 \quad \forall u \in [-1, 1] \quad |T_m(u)| \leq 1$

(ii) $x^m = \sum_{k=-m}^m q_m(k) T_k(x)$. "polynomial identity"

Proof: (i) choose θ p.t. $\cos \theta = u$ then $|T_m(u)| = |T_m(\cos \theta)| = |\cos(m\theta)| \leq 1$.

(ii) It suffices to prove the equality for $x = \cos \theta$, $\theta \in \mathbb{R}$.

$$x^m = \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)^m = \sum_{l=0}^m \underbrace{\frac{1}{2^m} \binom{m}{l}}_{= q_m(2l-m)} e^{i\theta(2l-m)}$$

$$= \sum_{|k| \leq m} q_m(k) e^{i\theta k}$$

$$= \sum_{|k| \leq m} q_m(k) \frac{e^{i\theta k} + e^{-i\theta k}}{2} \quad [\text{sym. } q_m(-k) = q_m(k)]$$

$$= \sum_{|k| \leq m} q_m(k) T_{ik}(x)$$

3. PROOF OF THE THEOREM.

Lemma: Let $S : \ell_\infty \rightarrow \ell_\infty$ self-adjoint bounded ($\|S\|_2 < \infty$), let $Q \in \mathbb{R}[\cos]$.

Then

$$\|Q(S)\|_2 \leq \max_{u \in [-\|S\|_2, \|S\|_2]} |Q(u)|.$$

Proof: (first proof). Use the spectral theorem for bounded self-adjoint operators (see [Rudin, Functional analysis, p. 308 / 303]) we have

$$\langle Q(S)f, Q(S)g \rangle = \int_{-\|S\|_2}^{\|S\|_2} Q(u)^2 dE_{f,g}(u) \text{ where } dE_{f,g}(u)$$

is the spectral measure of S attached to f .

Second proof (with a reduction to finite dimension)

Consider the finite-dimensional space $\mathcal{E}_n = \{f \in \mathcal{P}_k : \text{Supp}(f) \subset B_n\}$ equipped with the inner product $\langle f, g \rangle_{\mathcal{E}_n} = \langle f, g \rangle$

Define $S_n : \mathcal{E}_n \rightarrow \mathcal{E}_n$ by setting

$$\forall f \in \mathcal{E}_n \quad (S_n f)(x) = (Sf(x)) \mathbb{1}_{x \in B_n}$$

S_n is a self-adjoint linear operator on the (finite dimensional) Euclidean space $(\mathcal{E}_n, \langle \cdot, \cdot \rangle_{\mathcal{E}_n})$. Indeed

$$\forall f, g \in \mathcal{E}_n \quad \langle S_n f, g \rangle_{\mathcal{E}_n} = \underset{\substack{g \in \mathcal{E}_n \\ \uparrow}}{\langle Sf, g \rangle} = \underset{\substack{\uparrow \\ S \text{ auto-adjoint}}}{\langle f, Sg \rangle} = \underset{\substack{\uparrow \\ f \in \mathcal{E}_n}}{\langle f, S_n g \rangle_{\mathcal{E}_n}}.$$

Therefore there exists an orthonormal basis $(\varphi_1, \dots, \varphi_L)$ of \mathcal{E}_n made of eigenvectors of S_n . Writing λ_l for the eigenvalue associated to φ_l , we obtain for every $f \in \mathcal{P}_k$ and n large,

$$Q(S_n)f = Q(S_n) \left(\sum_{l \leq L} \langle f, \varphi_l \rangle \varphi_l \right) \\ = \sum_{l \leq L} \langle f, \varphi_l \rangle Q(\lambda_l) \varphi_l$$

$$\text{Hence } \|Q(S_n)f\|_2^2 = \sum_l \langle f, \varphi_l \rangle^2 Q(\lambda_l)^2 \leq \left(\max_{|\lambda| \leq \|S_n\|_2} |Q(\lambda)| \right)^2 \|f\|_2^2,$$

which implies $\|Q(S_n)\|_2 \leq \max_{|\lambda| \leq \|S_n\|_2} |Q(\lambda)|$.

and the proof follows from the fact that $\|Q(S_n)f\|_2 \xrightarrow{n \rightarrow \infty} \|Q(f)\|_2$ and $\|S_n\|_2 \leq \|S\|_2$.

(See exercise 1 and 2)

Proof of Van der Corput's bound:

By Property(ii) of the Chebyshev's polynomials, we have

$$\text{for every } m \geq 1 \quad \left(\frac{P}{\|P\|} \right)^m = \sum_{|k| \leq m} q_m(k) T_k \left(\frac{P}{\|P\|} \right).$$

key observation: If $d(x, y) > \|P\|$ $\langle P^m \mathbb{1}_x, \mathbb{1}_y \rangle = 0$

and therefore $\langle T_k \left(\frac{P}{\|P\|} \right) \mathbb{1}_x, \mathbb{1}_y \rangle = 0 \quad \forall |k| \leq d(x, y)$

For every $x, y \in V$, $m \geq 1$

$$\begin{aligned} P_m(x, y) &= \langle P^m \mathbb{1}_x, \mathbb{1}_y \rangle \\ &= \|P\|_2^m \langle \left(\frac{P}{\|P\|_2} \right)^m \mathbb{1}_x, \mathbb{1}_y \rangle \\ &= \|P\|_2^m \sum_{|k| \leq m} q_m(k) \underbrace{\langle T_k \left(\frac{P}{\|P\|_2} \right) \mathbb{1}_x, \mathbb{1}_y \rangle}_{= 0 \text{ if } |k| < d(x, y)} \\ &\leq \|P\|_2^m \sum_{d(x, y) \leq |k| \leq m} q_m(k) \underbrace{\|T_k \left(\frac{P}{\|P\|_2} \right)\|_2}_{\leq \max_{\omega \in [-1, 1]} |T_k(\omega)| \leq 1} \\ &\leq 2 \|P\|_2^m \csc p \left(- \frac{d(x, y)^2}{m} \right). \end{aligned}$$

4 ONE APPLICATION

The Varopoulos -Carne bound provides useful upper bounds on the speed at which the random walk moves away from 0. Write $|X_m| = d(0, X_m)$. When the graph G has sub-exponential volume growth, we obtain substantial improvements to the trivial bound $|X_m| \leq m$. In particular we have the following bounds when the growth is polynomial or stretch exponential:

Corollary (to Varopoulos -Carne bound)

(i) If G has polynomial volume growth ($\exists A, B_m \in A, n^D$) then $\exists C < \infty$, s.t.

$$\limsup_{m \rightarrow \infty} \frac{|X_m|}{\sqrt{m \log n}} \leq C \quad P_0\text{-a.s.}$$



(ii) If $|B_m| \leq A e^{Cm^\alpha}$ for some $A, C > 0$ and $0 < \alpha \leq 1$, then

$$\limsup_{m \rightarrow \infty} \frac{|X_m|}{m^{1/(2-\alpha)}} \leq (2C)^{1/(2-\alpha)} \quad P_0\text{-a.s.}$$

Proof: (i) Assume $|B_m| \leq A m^0$. Let $N = C \sqrt{m \log n}$

$$\begin{aligned} P[|X_m| \geq N] &\leq \sum_{k \geq N} P[|X_m| = k] \\ &\leq \sum_{k \geq N} |B_k| \cdot 2 \exp\left(-\frac{k^2}{2m}\right) \end{aligned}$$

$$\leq 2A \underbrace{\sum_{k \geq N} k^0 \exp\left(-\frac{k^2}{2m}\right)}$$

$$\leq C_1 N^{A+\epsilon} \exp\left(-\frac{N^2}{2m}\right)$$

$$\text{Hence } P\left[|X_n| \geq C\sqrt{n \log n}\right] \leq 2AC_1 \times (n \log n)^{\frac{D+1}{2}} n^{-\frac{C^2}{2}}$$

$$\text{If } C > \sqrt{D+1} \quad \sum_m P\left[\frac{|X_m|}{\sqrt{n \log n}} \geq C\right] < \infty$$

And Borel-Cantelli Lemma concludes that

$$\limsup_{n \rightarrow \infty} \frac{|X_n|}{\sqrt{n \log n}} \leq C \text{ a.s.}$$

(ii) exercice.

Remarks:

- In (i), it is possible to choose $C = \sqrt{D+1}$ (exercice)
- The bound (i) is not sharp for $G = \mathbb{Z}^d$. Indeed by the law of the iterated logarithm

$$\limsup_{n \rightarrow \infty} \frac{|X_n|}{\sqrt{n \log \log n}} < \infty$$

CHAPTER 5 :

RECURRENCE / TRANSIENCE

Ref: [LYONS - PERES, section 6.7] [WOESS, Theorem 3.24]

$G = (V, E)$ transitive, loc. finite, connected, infinite graph.

degree d , fixed origin $\sigma \in V$.

In this chapter we will obtain general bounds on $p_n(\sigma, \sigma)$ depending on the "isoperimetric profile" of the graph. As a consequence we obtain the following theorem, which determines for which graphs the SRW is recurrent / transient.

Theorem:

- (i) If the growth of G is at most quadratic (ie $\exists C: |B_m| \leq Cm^2$), then the SRW is recurrent.
- (ii) If the growth of G is at least cubic (ie $\exists c > 0: |B_m| \geq cm^3$), then the SRW is transient.

Remark: By Grigor'yan-Troliumov Theorem, any transitive satisfies either (i) or (ii). Therefore the theorem above provides us with a geometric characterization of recurrence / transience.

1. PROOF OF (i)

There are several ways to prove this statement, see e.g. [LYONS - PERES, exercise 2.86] for a proof using Nash-Williams criterion. Here we will prove it using Varopoulos-Carne's bound.

Proof of (i).

Let C_0 s.t. $\forall m \quad |B_m| \leq C_0 m^2$

By the corollary of Vanopoulos-Carne bound, we have for m large enough.

$$\frac{1}{2} \leq P_0 [X_{2m} \in B_N] \quad \text{where } N = \sqrt{C_0 \log m} \quad (C > 0 \text{ constant})$$

$$\leq \sum_{x \in B_N} P_0 [X_{2m} = x]$$

$$\leq |B_N| p_{2m}(0,0)$$

$$\leq C_0 \cdot C_0 \log m p_{2m}(0,0)$$

Hence $p_{2m}(0,0) \geq \frac{C_0 \cdot C_0}{m \log m}$, which implies that $\sum_{k \geq 0} p_k(0,0) = +\infty$.

Therefore the SRW is recurrent. ■

2 EXPANSION PROFILE.

Def: The expansion profile of \mathbb{G} is the function φ defined by

$$\forall u \geq 1 \quad \varphi(u) = \inf_{1 \leq |S| \leq u} \left(\frac{|S|}{|B_S|} \right)$$

Rk: φ is non increasing and $\lim_{u \rightarrow \infty} \varphi(u) = \Phi$ isoperimetric constant.

In particular (\mathbb{G} is amenable) $\hookrightarrow (\lim \varphi = 0)$.

Thm: Define for every $m \geq 1$ $R(m) = \min \{n : |B_n| \geq m\}$.

There exists a constant $c > 0$ s.t.

$$\forall u \geq 1 \quad \varphi(u) \geq \frac{c}{R(2u)}$$

Examples: If $G = \mathbb{Z}^d$, then we get $R(m) \approx m^{1/d}$
and we obtain for every $S \subset \mathbb{Z}^d$ finite

$$\frac{|\partial S|}{|S|} \geq \frac{c}{|S|^{1/d}} \quad \text{ie } |\partial S| \geq c |S|^{\frac{d-1}{d}}.$$

(The bound is sharp, up to constant)

- If G has exponential volume growth then $R(m) \approx \log m$

$$\forall S \subset V \quad \frac{|\partial S|}{|S|} \geq \frac{c}{\log(S)}$$

(The bound is not sharp if the graph is non amenable).

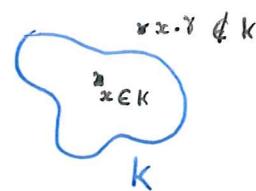
Proof: Case $G = (V, E) = \text{Cay}(F, S)$

The general case will be treated later using the mass-transport principle.

Let $K \subset V$ finite non empty. We will prove that

$$\frac{|\partial K|}{|K|} \geq \frac{1}{2R(2|K|)}$$

For $\gamma \in F$ let $A_\gamma = \{x \in K : x \cdot \gamma \notin K\}$



First observe that for $s \in S$ (generating set)

$$|A_s| = |\{x \in K : x \cdot s \notin K\}| \leq |\partial S|.$$



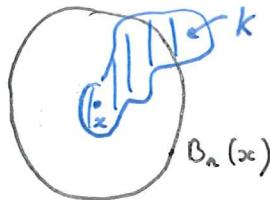
Since $A_{\gamma \cdot s} \subset \underbrace{\{x \in K : x \cdot \gamma \notin K\}}_{A_\gamma} \cup \underbrace{\{x \in F : x \cdot \gamma \in K, x \cdot \gamma \cdot s \notin K\}}_{A_{s \cdot \gamma^{-1}}}$

we have $\forall s \in S \quad |A_{\gamma \cdot s}| \leq |A_\gamma| + |A_{s \cdot \gamma^{-1}}|$

By induction, if $\gamma = s_1 \dots s_n$, we obtain $|A_\gamma| \leq n |\partial S|$.

Now, pick $\gamma \sim \text{Uniform}(B_n)$ where $n = R(2|k|)$.

Since $|B_n| \geq 2|k|$, we have $\forall x \in K \quad \mathbb{P}[x, \gamma \in K] = \frac{|B_n(x) \cap K|}{|B_n|} \leq \frac{|K|}{|B_n|} \leq \frac{1}{2}$.



Furthermore $n \cdot |\partial K| \geq \mathbb{E}[|A_\gamma|] = \mathbb{E}\left[\sum_{x \in K} \mathbb{1}_{x, \gamma \notin K}\right] = \sum_{x \in K} \underbrace{\mathbb{P}[x, \gamma \notin K]}_{\geq \frac{1}{2}}$.

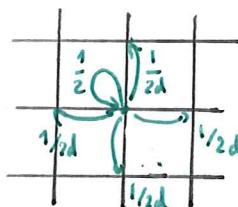
Therefore $\frac{|\partial K|}{|K|} \geq \frac{1}{2n} = \frac{1}{2R(2|k|)}$. ■

3 LAZY RANDOM WALK

Def: The Lazy random walk (Lazy RW) on G starting at x_0 is the homogeneous Markov chain $(Y_n)_{n \geq 0}$ with

- state space V
- initial state $Y_0 = x$
- transition probabilities

$$q(x, y) = \begin{cases} \frac{1}{2} & \text{if } x = y \\ \frac{1}{2} p(x, y) & \text{if } x \neq y. \end{cases}$$



"a slower version of
the SRW"

main advantage: the Lazy RW is always aperiodic.

Notation: • $Q := \frac{1}{2} (I + P)$ operator associated to the Lazy RW.

• $q_m(x, y) = \mathbb{P}[Y_m = y | Y_0 = x]$ the m -step transition probabilities

Rk: $q_m(x, y) = \langle Q^m \mathbb{1}_x, \mathbb{1}_y \rangle = \frac{1}{2^n} \sum_{k=0}^m \binom{m}{k} p_k(x, y)$

We will use the following properties of the Lazy RW:

Properties:

$$(i) \forall n \geq 0 \quad p_{2n}(0,0) \leq 2 q_{2n}(0,0). \quad [\text{Comparison with SRW}]$$

$$(ii) \forall m, n \geq 0 \quad q_{m+n}(x, z) = \sum_{y \in V} q_m(x, y) q_n(y, z) \quad [C-K]$$

Proof: (ii) is a standard property of Markov chains.

(i) First observe that $(p_{2k}(0,0))$ is decreasing in k .

$$\text{Indeed } p_{2k+2}(0,0) = \sum_y p_2(0,y) p_{2k}(y,0) \leq p_{2k}(0,0) \underbrace{\sum_y p_2(0,y)}_{=1}.$$

$$\text{Then } q_{2n}(0,0) = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n}{2k} p_{2k}(0,0)$$

$$\geq p_{2n}(0,0) \underbrace{\frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n}{2k}}_{= \frac{1}{2}}$$

Rk: By (i), we have (Lazy RW transient) \Rightarrow (SRW transient).

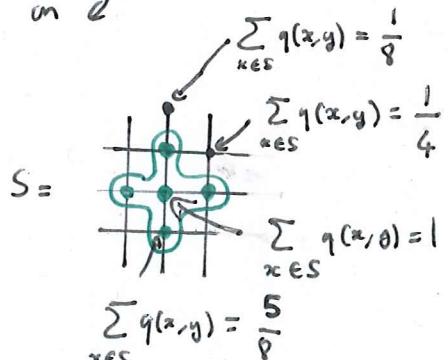
In fact, it is easy to see that it is an equivalence (see exercises).

4. EVOLVING SETS.

Def: For $S \subset V$ finite and $u \in [0,1]$ we define the set

$$F(S, u) = \{y \in V : \sum_{x \in S} q(x, y) \geq u\}$$

Example on \mathbb{Z}^2



$$F(S, u) = \begin{cases} \text{[] } & \text{if } u > \frac{5}{8} \\ \text{[] } & \text{if } \frac{1}{4} < u \leq \frac{5}{8} \\ \text{[] } & \text{if } \frac{1}{8} < u \leq \frac{1}{4} \\ \text{[] } & \text{if } 0 \leq u < \frac{1}{8} \end{cases}$$

Rk: If $u \leq \frac{1}{2}$ then $F(S, u) \supset S$ and $F(S, u) \setminus S \subset \overbrace{\{y \in S^c : \exists x \in S, xy\}}^{\text{"outer boundary"}}$

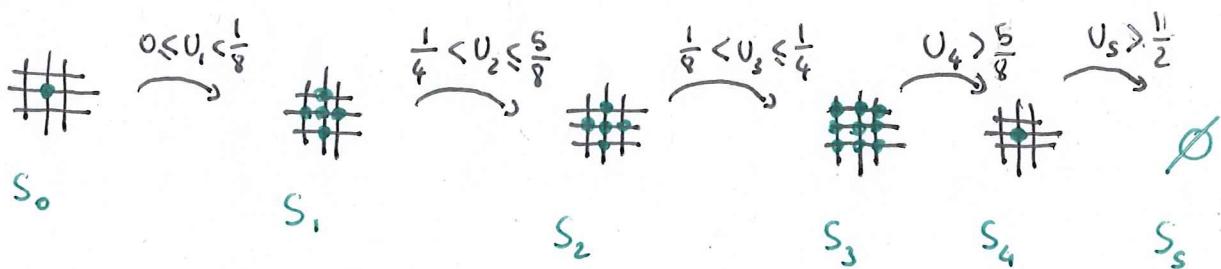
If $u > \frac{1}{2}$ then $F(S, u) \subset S$ and $S \setminus F(S, u) \subset \overbrace{\{x \in S : \exists y \in S^c, xy\}}^{\text{"inner boundary"}}$

Def: Let U_1, U_2, \dots be a sequence of iid uniform variables in $[0, 1]$.

Define the Markov Chain $(S_n)_{n \in \mathbb{N}}$ by setting

- $S_0 = \{0\}$
- $S_{n+1} = F(S_n, U_{n+1})$.

Example on \mathbb{Z}^2



Rk: \emptyset is an absorbing state.

Properties: -

(i) For every n we have $q_n(0, \infty) = \mathbb{P}[\infty \in S_n]$.

(ii) $|S_n|$ is a martingale (for the filtration generated by U_1, \dots, U_n)

Remark: We have $q_n(0, 0) = \mathbb{P}[S_n \neq \emptyset]$ and $q_n(0, 0) \xrightarrow{n \rightarrow \infty} 0$ (exercise)

Therefore $S_n \rightarrow \emptyset$ a.s.

Proof: (i) By induction on n . First, we have $q_0(0, \infty) = \mathbb{P}_{x=0}[S_0 \ni x]$.
Then for $n \geq 0$, we have

$$\begin{aligned} q_{n+1}(0, \infty) &= \sum_{y \in V} q_n(0, y) q(y, \infty) \stackrel{(IH)}{=} \sum_{y \in V} \mathbb{P}[y \in S_n] q(y, \infty) \\ &= \mathbb{E} \left[\sum_{y \in S_n} q(y, \infty) \right]. \end{aligned}$$

$$\begin{aligned} \text{Since } \sum_{y \in S_m} q(y, x) &= \mathbb{P}[U_{m+1} \leq \sum_{y \in S_m} q(y, x) | S_m] \\ &= \mathbb{P}[x \in S_{m+1} | S_m], \end{aligned}$$

we finally get $q_{m+1}(0, x) = \mathbb{P}[x \in S_{m+1}]$

$$\begin{aligned} \text{(ii)} \quad E[|S_{m+1}| | S_m] &= \sum_{y \in V} \mathbb{P}[y \in S_{m+1} | S_m] \\ &= \sum_{y \in V} \sum_{x \in S_m} q(x, y) \\ &= \sum_{x \in S_m} \underbrace{\sum_{y \in V} q(x, y)}_{=1} = |S_m| \end{aligned}$$

5. TRANSIENCE FOR GRAPHS OF AT LEAST CUBIC GROWTH.

In this section we prove the following theorem.

Theorem:

Let φ be the expansion profile of G . Define $\varepsilon_m > 0$ by

$$\int_4^{\varepsilon_m} \frac{8d^2 \cdot du}{u \varphi(u)} = m.$$

Then we have $q_{2m}(0, 0) \leq \varepsilon_m$.

Before proving let us see how it implies (ii) in the main theorem of the chapter. Assume G has at least cubic growth. The theorem in Section 2 implies that $\exists c > 0$ s.t.

$$\forall u \geq 1 \quad \varphi(u) \geq \frac{c}{u^{1/3}}$$

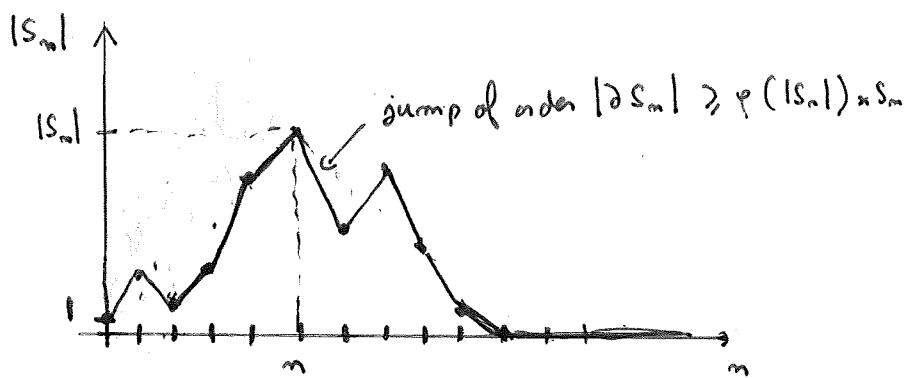
and the theorem above concludes that $\exists c' > 0$ s.t.

$$\forall m \geq 1 \quad q_{2m}(0, 0) \leq \frac{c'}{m^{3/2}}.$$

Therefore $\sum_{m \geq 1} p_m(0, 0) < \infty$ and the SRW is transient.

Idea of proof:

We want to bound $p_\infty(0,0) = \mathbb{P}[|S_\infty|=0]$ where $|S_n|$ is martingale satisfying $|S_{n+1}| = |S_n| + z_{n+1}$ where z_m is a centered r.v. of order $\sqrt{\mathbb{E}[z_m^2]} \approx |\partial S_n| \geq \varphi(|S_n|) \approx s_m$. Roughly, $|S_n|$ is "faster" than the martingale M_n defined by $\begin{cases} M_0 = 1 \\ M_{n+1} = M_n + \varepsilon_{n+1} \varphi(M_n) \eta_n \end{cases}$ where ε_n iid $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$.



↳ "reduction to a 1-D martingale problem". In order to get an estimate at which speed s_m gets absorbed at 0, we rather consider the supermartingale $\sqrt{|S_n|}$ which is "drifted" toward 0. ($\sqrt{|S_n|}$ is a super-martingale by Jensen inequality).

Introduce $S_m := E[\sqrt{|S_m|}]$.

$$\text{Rk: } S_m \leq E[|S_m|]^{\frac{1}{2}} = 1.$$

Lemma 1: For all $m \geq 0$ we have

$$q_{2m}(0,0) \leq S_m^2$$

Proof: Using Chapman-Kolmogorov for the lazy RW and reversibility, we find

$$q_{2m}(0,0) = \sum_{x \in V} q_m(0,x)^2 = \sum_{x \in V} P[x \in S_m]$$

Introduce Σ_n , an independent copy of S_n .

$$\begin{aligned}
 q_{2n}(0,0) &= \sum_{x \in V} \mathbb{P}[x \in S_n, x \in \Sigma_n] \\
 &= \mathbb{E} \left[\sum_{x \in V} \mathbb{1}_{x \in S_n} \mathbb{1}_{x \in \Sigma_n} \right] \\
 &= \mathbb{E}[|S_n \cap \Sigma_n|] \\
 &\leq \mathbb{E}[|S_n| \wedge |\Sigma_n|] \\
 &\leq \mathbb{E}[\sqrt{|S_n| |\Sigma_n|}] = s_n^2 \quad \blacksquare
 \end{aligned}$$

Lemma 2.

For every S s.t. $\mathbb{P}[S_n = S] > 0$ and $S \neq \emptyset$, we have

- $\mathbb{E}[|S_{n+1}| \mid S_n = S, U_{n+1} < \frac{1}{2}] = |S| + \frac{1}{d} |\partial S|$, and
- $\mathbb{E}[|S_{n+1}| \mid S_n = S, U_{n+1} \geq \frac{1}{2}] = |S| - \frac{1}{d} |\partial S|$.

Proof: Since (S_n) is a martingale, the second item follows from the first one.

For $y \in V$ we have

$$\begin{aligned}
 \mathbb{P}[y \in S_{n+1} \mid S_n = S, U_{n+1} < \frac{1}{2}] &= \mathbb{P}[U_{n+1} \leq \sum_{x \in S} q(x,y) \mid U_{n+1} < \frac{1}{2}] \\
 &= \begin{cases} 1 & \text{if } y \in S \quad (\text{by laziness}) \\ 2 \underbrace{\sum_{x \in S} q(x,y)} & \text{if } y \notin S \end{cases} \\
 &= \frac{1}{d} \sum_{x \in S} \mathbb{1}_{x \sim y}
 \end{aligned}$$

Therefore,

$$\mathbb{E}[|S_{m+1}| \mid S_m = S, U_{m+1} < \frac{1}{2}] = |S| + \frac{1}{d} \sum_{y \notin S} \sum_{x \in S} \mathbb{I}_{x \sim y} = |S| + \frac{1}{d} |\partial S| \blacksquare$$

Lemma 3: For every $m \geq 0$, we have -

$$\mathbb{E}[\sqrt{|S_m|} \mid S_m] \leq (1 - \Psi(|S_m|)) \sqrt{|S_m|} \text{ where } \Psi(u) := \frac{1}{8d^2} \varphi(u)^2$$

Proof: We will use the following elementary inequality.

$$\forall t \in [0, 1] \quad \frac{\sqrt{1+t}}{2} + \frac{\sqrt{1-t}}{2} \leq 1 - \frac{t^2}{8}. \text{ (exercise).}$$

Let $S \subseteq V$ s.t. $\mathbb{P}[S_m = S] > 0$ and $S \neq \emptyset$

$$\mathbb{E}[\sqrt{|S_{m+1}|} \mid S_m = S] = \frac{1}{2} \mathbb{E}[\sqrt{|S_m|} \mid S_m = S, U_{m+1} < \frac{1}{2}] + \frac{1}{2} \mathbb{E}[\sqrt{|S_m|} \mid S_m = S, U_{m+1} \geq \frac{1}{2}]$$

$$\stackrel{\text{Jensen}}{\leq} \frac{1}{2} \sqrt{|S| + \frac{1}{d} |\partial S|} + \frac{1}{2} \sqrt{|S| - \frac{1}{d} |\partial S|}$$

* Lemma 2

$$= \sqrt{|S|} \left(\frac{1}{2} \sqrt{1 + \frac{1}{d} \frac{|\partial S|}{|S|}} + \frac{1}{2} \sqrt{1 - \frac{1}{d} \frac{|\partial S|}{|S|}} \right)$$

$$\leq \sqrt{|S|} \left(1 - \frac{1}{8d^2} \left(\frac{|\partial S|}{|S|} \right)^2 \right)$$

$$\leq \sqrt{|S|} \left(1 - \underbrace{\frac{1}{8d^2} \varphi(|S|)^2}_{\varphi(|S|)} \right)$$

$$= \Psi(|S|)$$

Proof of the theorem.

$$S_m = E[\sqrt{|S_m|}] \stackrel{?}{=} E[\sqrt{|S_m|} (1 - \Psi(|S_m|))] = S_m - E[\underbrace{\sqrt{|S_m|}}_{\text{Lemma 3}} \Psi(|S_m|)]$$

$\nearrow \text{im } |S_m| \quad \searrow \text{im } |S_m|$

Observation: $1 = E[|S_m|] = \sum_{S \in V} |S| P[S_m = S]$

We introduce a size-biased version of S_m , defined by

$$P[S_m^* = S] = |S| P[S_m = S]$$

This way, we have for every f measurable

$$E[f(S_m) | S_m] = E[f(S_m^*)].$$

Using this new variable, we obtain $S_m = E\left[\frac{1}{\sqrt{|S_m^*|}}\right]$

$$E[\sqrt{|S_m|} \Psi(|S_m|)] = E\left[\frac{1}{\sqrt{|S_m^*|}} \Psi(|S_m^*|)\right]$$

$$\geq E\left[\frac{1}{\sqrt{|S_m^*|}} \Psi(|S_m^*|) \middle| \frac{1}{\sqrt{|S_m^*|}} > \frac{S_m}{2}\right]$$

$$\leq \frac{4}{S_m^2}$$

$$\geq \Psi\left(\frac{4}{S_m^2}\right) E\left[\frac{1}{\sqrt{|S_m^*|}} \middle| \frac{1}{\sqrt{|S_m^*|}} > \frac{S_m}{2}\right]$$

$$\geq \Psi\left(\frac{4}{S_m^2}\right) \cdot \frac{1}{2} S_m.$$

For the last inequality we use that for a positive integrable random variable X , $E[X \mathbb{1}_{X \geq \frac{1}{2} E[X]}] = E[X] - E[X \mathbb{1}_{X < \frac{1}{2} E[X]}] \geq \frac{1}{2} E[X]$.

Finally, we obtain $\forall m \geq 0$

$$\delta_{m+1} \leq \delta_m \left(1 - \frac{1}{2} \psi\left(\frac{4}{\delta_m^2}\right)\right).$$

This implies

$$\frac{\delta_{m+1}}{\delta_m} \leq e^{-\frac{1}{2} \psi\left(\frac{4}{\delta_m^2}\right)}$$

$$\text{i.e. } -\log(\delta_{m+1}) + \log(\delta_m) \geq \frac{1}{2} \psi\left(\frac{4}{\delta_m^2}\right)$$

which gives (using $\delta_{k+1} \leq \delta_k$ and ψ is decreasing) for every k

$$\int_{\delta_{k+1}}^{\delta_k} \frac{2 dt}{t \psi(4/t^2)} \geq \frac{2}{\psi(4/\delta_{k+1}^2)} \int_{\delta_{k+1}}^{\delta_k} \frac{dt}{t} \geq 1$$

Summing over $k=0, \dots, m-1$, we get.

$$\int_{\delta_m}^{\delta_0} \frac{2 dt}{t \psi(4/t^2)} \geq m$$

By using the change of variable $u = 4/t^2$, we finally get

$$\int_4^{4/\delta_m^2} \frac{du}{u \psi(u)} \geq m = \int_4^{4/\varepsilon_m} \frac{du}{u \psi(u)}$$

Hence $\delta_m^2 \leq \varepsilon_m$ and Lemma 1 concludes that

$$q_{2m}(0,0) \leq \delta_m^2 \leq \varepsilon_m$$

1. ENTROPY.

Def: Let \mathcal{X} be a finite set and X a r.v. with values in \mathcal{X} .

The Shannon entropy of X is defined by

$$H(X) = - \sum_{x \in \mathcal{X}} P[X=x] \log(P[X=x]).$$

Ref: See Yadin "Harmonic Functions on groups", Sect. 4.3.

, Cover and Thomas "Elements of information theory".

, blog "Yoo Box" → see "the town".

Intuition: $H(X)$ ≈ expected number of bits needed to encode X

≈ \log ("size of the set where X lives")

Properties: • $H(X) \leq \log(|\mathcal{X}|)$ with equality iff $X \sim \text{Uniform}(\mathcal{X})$

• If X r.v. in \mathcal{X} and Y r.v. in \mathcal{Y} (with $|\mathcal{X}| < \infty$ and $|\mathcal{Y}| < \infty$)

$H(X, Y) \leq H(X) + H(Y)$ with equality iff X, Y indep.

• $H(X) \leq H(X, Y)$ with equality iff $Y \in \sigma(X)$.

Entropy of the SRW

$$\text{Define } H(x_m) = - \sum_{y \in V} P_x[x_m=y] \log(P_x[x_m=y]).$$

Rk: Independent of the starting point x (by invariance).

Prop: For $m, n \geq 0$, we have $H(x_{m+n}) \leq H(x_m) + H(x_n)$.

$$\begin{aligned}
 \text{Proof: } H(X_{m+n}) &\leq H(X_m, X_{m+n}) \\
 &= -\sum_{x \in V} P_o[X_m = x] \sum_{y \in V} P_o[X_{m+n} = y | X_m = x] \\
 &\quad \times \log(P_o[X_{m+n} = y | X_m = x]) \\
 &= -\sum_{x \in V} P_o[X_m = x] \underbrace{\sum_{y \in V} P_o[X_{m+n} = y | X_m = x] \log(P_o[X_{m+n} = y | X_m = x])}_{\substack{\text{(Markov)} \\ = H(X_m)}} \\
 &\quad - \sum_{x \in V} P_o[X_m = x] \underbrace{\sum_{y \in V} P_o[X_{m+n} = y | X_m = x] \log(P_o[X_m = x])}_{= 1} \\
 &= H(X_m) + H(X_{m+n})
 \end{aligned}$$

As a consequence, by Fekete's Lemma, $\frac{H(X_n)}{n}$ converges in $[0, \infty)$.

Def: The Ave₂ entropy of the SRW is defined by

$$h = \lim_{n \rightarrow \infty} \frac{H(X_n)}{n}$$

Intuition: "If $h > 0$, X_m lives in set of size $\approx e^{hn}$ "

Thm: For every $n \geq 0$, we have

$$(i) \quad H(X_n) \leq P \log n + \log(d) E[|X_n|],$$

$$(ii) \quad n^2 \log\left(\frac{1}{P}\right) + E[|X_n|^2] \leq n(H(X_n) + \log 2).$$

Corollary: $P > 0$ if and only if $h > 0$.

Proof \Rightarrow follows from (i)

\Leftarrow follows from $E[|X_n|]^2 \leq E[|X_n|^2]$ and (ii).

CHAPTER 7:

SPEED AND LIOUVILLE PROPERTY.

Ref.: [LYONS-PERES, Chap. 14]

$G = (V, E)$ transitive, loc. finite, connected, infinite graph,
 degree d , fixed origin $o \in V$
 volume growth exponent v ($|B_n| = e^{vn + o(n)}$)
 spectral radius s ($P_n(0, 0) = s^{2n + o(n)}$)

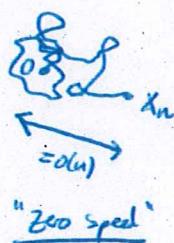
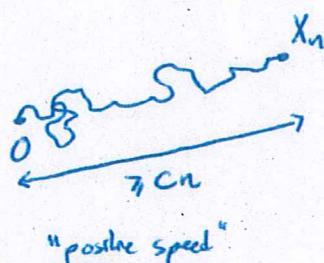
Goal: Understand asymptotic behaviour of $\frac{E[|X_n|]}{n}$.

- If G has subexponential volume growth, then the Varopoulos-Carne bound implies $\lim_{n \rightarrow \infty} \frac{E[|X_n|]}{n} = 0$. (ex.) "zero speed"

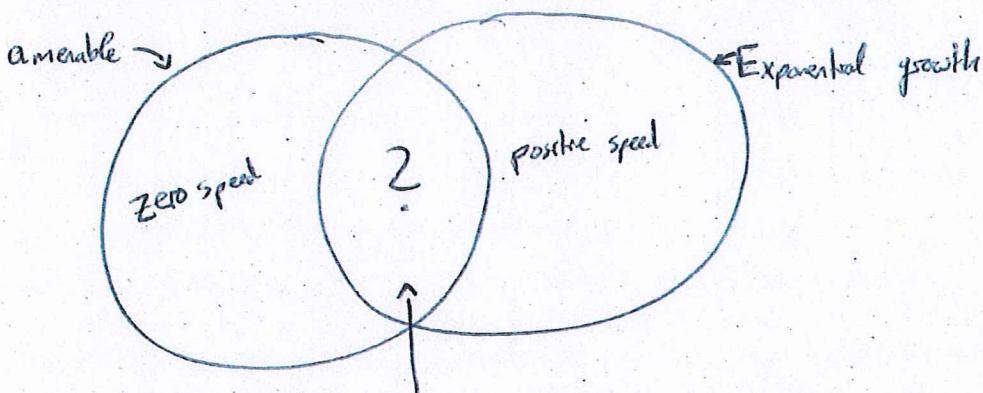
- If G is non-amenable, then $P_n(0, x) \leq s^n$ with $s < 1$.

So $P_o[X_n \in B_n] \leq |B_n| \cdot s^n = e^{avn + o(n)} s^n \xrightarrow{n \rightarrow \infty} 0$ if $e^{av} s < 1$.

$\Rightarrow \liminf_{n \rightarrow \infty} \frac{|X_n|}{n} \geq \frac{\log(\frac{1}{s})}{v} > 0$ a.s. "positive speed"



Summary:



$LL(\mathbb{Z})$ is an example of an amenable graph with exponential growth.

1. SPEED OF THE SRW. AND ENTROPY

Write $|X| = d(0, x)$ for every $x \in V$.

We have $\mathbb{E}_0[|X_{n+m}|] \leq \mathbb{E}_0[|X_n|] + \underbrace{\mathbb{E}_0[d(X_n, X_{n+m})]}_{\Delta-\text{eq.}}$

$= \mathbb{E}_0[|X_n|]$ by Markov property at invariance

By Fekete's subadditivity lemma, $\frac{\mathbb{E}_0[|X_n|]}{n}$ converges in $[0, +\infty]$.

Def.: The speed of the SRW. is defined by

$$\ell = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_0[|X_n|]}{n} = \inf_{n \geq 0} \frac{\mathbb{E}_0[|X_n|]}{n}$$

Examples:

- $\ell = 0$ if G has subexponential growth.
- $\ell \geq \frac{\log(\frac{1}{2}/s)}{\sqrt{s}} > 0$ if G is non-amenable.
- $\ell = \frac{d-2}{d}$ if G is a d -regular tree. ($|X_n|$ increases by $+1$ with prob. $\frac{d-1}{d}$, decreases by -1 with $\frac{1}{d}$)

Proposition:

$$\frac{|X_n|}{n} \xrightarrow{n \rightarrow \infty} \ell \quad \text{a.s.}$$

Proof: Omit. Follows from an application of Kingman's subadditive ergodic theorem (see Thm. 14.10. in [LYONS-PERES]).

Thm.:

$$\ell > 0 \text{ if and only if } h > 0$$

Recall Avez entropy $h = \lim_{n \rightarrow \infty} \frac{H(X_n)}{n}$.

Proof:

$$\begin{aligned} \Rightarrow H(X_n) &= H(X_n | X_0) = \underbrace{H(|X_n|)}_{\leq \log(n)} + \underbrace{H(X_n | |X_n|)}_{\leq \log(d)} \leq \log(n) + \log(d) \cdot \mathbb{E}[|X_n|] \\ &\leq \log(n) \leq \mathbb{E}[\log B_{|X_n|}] \leq d^{|X_n|} \end{aligned}$$

Divide by n and take limit to conclude.

By Varopoulos - Carre bound,

$$P_0[X_n=x] \leq 2^{-n} \exp\left(-\frac{|x|^2}{2^n}\right).$$

Hence,

$$\begin{aligned} H(X_n) &= \sum_x P_0[X_n=x] \lg\left(\frac{1}{P_0[X_n=x]}\right) \\ &\geq \sum_x P[X_n=x] \cdot \left(-\lg(2) - n \lg(2) + \frac{|x|^2}{2^n}\right) \\ &= -\lg(2) - n \lg(2) + \frac{1}{2^n} \underbrace{\mathbb{E}[|X_n|^2]}_{\geq \mathbb{E}[|X_n|]^2} \end{aligned}$$

Divide by n and take limit to get $H \geq \lg\left(\frac{1}{2}\right) + C^2$.



2. TAIL- σ -ALGEBRA AND ENTROPY

Def. For a SRW $(X_n)_{n \geq 0}$ starting at 0,

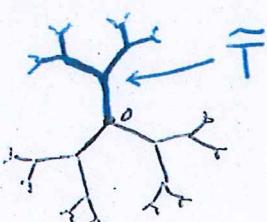
we define $\mathcal{T} := \bigcap_{n \geq 0} \sigma(X_n, X_{n+1}, \dots)$.

"tail- σ -algebra"

Def. We say that \mathcal{T} is trivial if $\forall A \in \mathcal{T} : P_0[A] \in \{0, 1\}$.

Example: $G = \mathbb{T}_3$ (3-regular tree)

Consider $\tilde{T} \subseteq \mathbb{T}_3$ defined by the diagram:



For $A = \{X_n \text{ escapes in } \tilde{T}\} = \bigcup_{m \geq 0} \{\forall k \geq m : X_k \in \tilde{T}\}$,

one can check that $A \in \mathcal{T}$ and $P_0[A] = \frac{1}{3}$.

\rightarrow " \mathcal{T} is not trivial for the tree."

Recall: Let $(F_n)_{n \in \mathbb{N}}$ be a decreasing family of σ -algebras,
i.e. $F_n \supseteq F_m$, $\forall n \geq m$.

Then $(E[Y|F_n])_{n \in \mathbb{N}}$ is a backwards martingale

and $E[Y|F_n] \xrightarrow{n \rightarrow \infty} E[Y|F_\infty]$ a.s. and in L^2 ,

where $F_\infty = \bigcap_{n \geq 0} F_n$.

(see [DURRETT], Probability: Theory and Examples, Thm. 5.6.3.)

Thm.:

$h > 0$ if and only if \mathcal{T} is not trivial.

Rk: Holds in more general setup (see [LYONS-PERES], Thm. 14.7),
in particular also for lazy RW.

Proof:

Step 1: \mathcal{T} is trivial $\Leftrightarrow H(X_1, \dots, X_k | \mathcal{T}) = H(X_1, \dots, X_k)$, $\forall k$.

If $H(X_1, \dots, X_k | \mathcal{T}) = H(X_1, \dots, X_k)$, $\forall k$, then $(X_1, \dots, X_k) \perp\!\!\!\perp \mathcal{T}$, $\forall k$.

So \mathcal{T} is trivial (exercise).

The other direction is clear.

Step 2: $h > 0 \Leftrightarrow H(X_1, \dots, X_k | \mathcal{T}) = H(X_1, \dots, X_k)$, $\forall k$.

By backwards martingale convergence ($n > k$),

$$H(X_1, \dots, X_k | X_n) = H(X_1, \dots, X_k | \sigma(X_n, X_{n+1}, \dots)) \xrightarrow{n \rightarrow \infty} H(X_1, \dots, X_k | \mathcal{T})$$

Markov property

So

$$\begin{aligned} H(X_1, \dots, X_k) - H(X_1, \dots, X_k | \mathcal{T}) &= \lim_{n \rightarrow \infty} H(X_1, \dots, X_k) - \underbrace{H(X_1, \dots, X_k | X_n)}_{\text{Markov property}} \\ &= H(X_1, \dots, X_k, X_n) - H(X_n) \\ &= H(X_1, \dots, X_k) + H(X_n | X_1, \dots, X_k) - H(X_n) \\ &\stackrel{\dagger}{=} \lim_{n \rightarrow \infty} H(X_n) - H(X_{n-k}) \\ &= \lim_{n \rightarrow \infty} \sum_{m=n-k+1}^n H(X_m) - H(X_{m-1}) \underset{\text{converges to } h}{=} k \cdot h \end{aligned}$$



3 HARMONIC FUNCTIONS

Def: We define the operator $\Delta : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by

$$\forall f \in \mathbb{R}^V \quad \forall x \in V \quad (\Delta f)(x) = \frac{1}{d} \sum_{y \sim x} (f(y) - f(x))$$

Rk: $(\Delta f)(x) = \underbrace{\left(\frac{1}{d} \sum_{y \sim x} f(y) \right)}_{\text{"average of } f \text{ on the neighbors of } x"} - f(x)$

"average of f on the neighbors of x "

Rk: The operator P can be defined on \mathbb{R}^V and

$$\Delta = P - \text{Id}$$

Def: Let $f : \mathbb{R} \rightarrow V$. We say that f is a harmonic function (HF) if $\Delta f = 0$

Rk: f harmonic $\Leftrightarrow \forall x \quad f(x) = \frac{1}{d} \sum_{y \sim x} f(y)$

Ex: • constants

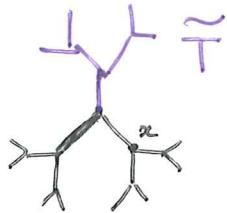
• $G = \mathbb{Z} \quad f : \mathbb{Z} \rightarrow \mathbb{R}$ harmonic

$$\Leftrightarrow \forall k \in \mathbb{Z} \quad f(k) - f(k-1) = f(k+1) - f(k)$$

$$\Leftrightarrow \exists a, b \in \mathbb{R} \quad \forall k \in \mathbb{Z} \quad f(k) = ak + b.$$

• $G = \mathbb{Z}^2 \quad f(x, y) = x^2 - y^2$

$$\bullet \quad G = \mathbb{T}^3$$



$$f(x) = P_x \left[\exists n_0 \geq 1 : \forall n \geq n_0, X_n \in \tilde{\Gamma} \right]$$

(ex: f is harmonic, non constant)

Prop: (Connection with the RW)

Let $h \in \mathbb{R}^V$. Let $(X_n)_{n \geq 0}$ be a SRW (or a L RW) from $x \in V$. The following are equivalent

(i) h is harmonic

(ii) $Ph = h$

(iii) $(h(X_n))_{n \geq 0}$ is a martingale (wrt the filtration

$$\mathcal{F}_n = \sigma(X_0, \dots, X_n))$$

Proof: (i) \Leftrightarrow (ii) $\Delta h = 0 \Leftrightarrow (P - \text{Id}) h = 0 \Leftrightarrow Ph = h$.

(ii) \Leftrightarrow (iii) We prove it for $(X_n)_n$ being the SRW from x

$$E_x [h(X_{n+1}) | X_0, \dots, X_n] = \sum_z 1_{X_n=z} E_x [h(X_{n+1}) | X_0, \dots, X_n]$$

$$= \sum_z 1_{X_n=z} E_z (h(X_1))$$

$$= \sum_z 1_{X_n=z} (Ph)(z)$$

$$= Ph(X_n) \quad \text{a.s.}$$

$$(iii) \Leftrightarrow \forall n \quad P_h(X_n) = h(X_n) \quad P_x\text{-a.s.}$$

$$\Leftrightarrow \forall x \quad P_h(x) = h(x)$$

■

Exercise: Check (ii) \Leftrightarrow (iii) for the LRW.

4 LIOUVILLE PROPERTY.

Def: We say that G has the Liouville property (LP) if every bounded harmonic function is constant.

Ex: \mathbb{Z} has LP. \mathbb{T}_3 has not LP.

Prop. Assume G is recurrent, then G has LP.

Proof: Let h be a bounded HF. Let $(X_n)_{n \geq 0}$ SRW from Θ .

Let $x \in V$ and $T_x := \min \{ n : X_n = x \}$.

By recurrence $T_x < \infty$ a.s. Since $h(X_n)$ is a bounded martingale we have

$$h(0) = E_0 [h(X_{T_x})] = E_0 [h(x)] = h(x)$$

Hence h is constant.

Corollary: \mathbb{Z}^2 has LP.

Prop. Assume that for every $x, y \in V$ neighbours, there exist $(Y_n)_{n \geq 0}, (\tilde{Y}_n)_{n \geq 0}$ coupled s.t.

- $(Y_n)_{n \geq 0}$ LRW from x , $(\tilde{Y}_n)_{n \geq 0}$ LRW from y
- $\mathbb{P}\{\exists n_0 > 0 \quad \forall n \geq n_0 \quad Y_n = \tilde{Y}_n\} = 1$.

Then G has LP.

Rk: a characterization exists. See [Yadin, "harmonic function on groups", sect. 4.6]

Proof: Let h bounded HF. Let $x \sim y$.

For every $n \geq 0$ we have $h(x) = \mathbb{E}[h(Y_n)]$
and $h(y) = \mathbb{E}[h(\tilde{Y}_n)]$.

$$\begin{aligned} |h(x) - h(y)| &= |\mathbb{E}[h(Y_n) - h(\tilde{Y}_n)]| \\ &\leq \mathbb{E}[|h(Y_n) - h(\tilde{Y}_n)|] \\ &\leq 2 \|h\|_\infty \mathbb{P}[Y_n \neq \tilde{Y}_n] \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Hence $\forall x \sim y \quad h(x) = h(y)$. h is constant. ■

Appli: $\mathbb{Z}^d, d \geq 1$ has LP.

Prop. LL(G) has LP $\Leftrightarrow G$ is recurrent.

Proof: (exercise)

⇒ Couple two LRW (Y_n, ξ_n) and $(\tilde{Y}_n, \tilde{\xi}_n)$.
↑ "position in G" ↗ "Lamp. configuration"

First couple (Y_n) and (\tilde{Y}_n) until they meet,
then match the lamps.

⇒ Consider $h(x) = P_{x0} \left[\begin{array}{l} \text{The Lamp at 0 is on} \\ \text{for all time } n \text{ sufficiently large} \end{array} \right]$.

Thm [Kaimanovitch - Vershik]

The following are equivalent

(i) $\ell > 0$

(ii) G does not have LP.

5 COUPLING BETWEEN LAZY WALKS

Def: We call Bernoulli walk (BW) a Markov Chain $(T_n)_{n \geq 0}$ with state space \mathbb{N} , initial state $T_0 = 0$ and transition probabilities given by

$$P_{xy} = \begin{cases} \frac{1}{2} & \text{if } y=x \text{ or } y=x+1, \\ 0 & \text{otherwise.} \end{cases}$$

Prop. Let $(X_n)_{n \geq 0}$: SRW from $x \in V$, $(T_n)_{n \geq 0}$ BW indep. of $(X_n)_{n \geq 0}$. Then

$$Y_n := X_{T_n}$$

defines a LRW from x .

Proof: Let $y_0, \dots, y_n \in V$. Define $t_0 = 0$ and by induction $t_{i+1} = \begin{cases} t_i & \text{if } y_{i+1} = y_i \\ t_i + 1 & \text{otherwise} \end{cases}$

$$\begin{aligned} & \mathbb{P}[Y_0 = y_0, \dots, Y_n = y_n] \\ &= \mathbb{P}[X_{t_0} = y_0, \dots, X_{t_n} = y_n, T_0 = t_0, \dots, T_n = t_n] \\ &= \left(\prod_{i=0}^{n-1} \left(\frac{1}{d} \mathbb{1}_{y_i = y_{i+1}} + \frac{1}{d} \mathbb{1}_{y_i \neq y_{i+1}} \right) \right) * \frac{1}{2^n} \\ &= \prod_{i=0}^{n-1} \left(\frac{1}{2} \mathbb{1}_{y_i = y_{i+1}} + \frac{1}{2d} P(y_i, y_{i+1}) \right) \end{aligned}$$

■

Prop: Let $x \in V$. There exist $(z_n)_{n \geq 0}, (\tilde{z}_n)_{n \geq 0}$ two LRW from x o.t.

$$\mathbb{P}[\exists n_0 : \forall n \geq n_0 z_{n+1} = \tilde{z}_n] = 1$$

Lemma: $\exists (T_n)_{n \geq 0}, (T'_n)_{n \geq 0}$ DW o.t.

$$\mathbb{P}[\exists n_0 : \forall n \geq n_0 T_{n+1} = T'_n] = 1$$

Pf of the lemma:

Let $(T_n), (T'_n)_n$ be two indep. RW

$$\bar{\tau} := \min \{ n : T_n' = T_{n+1} \}$$

$\bar{\tau} < \infty$ a.s. (because $(T'_n - T_n)$ is a LRU on \mathbb{Z} and therefore is recurrent)

$$\text{Define } \tilde{T}_n = \begin{cases} T'_n & n \leq \bar{\tau}, \\ T_{n+\bar{\tau}} & n \geq \bar{\tau}. \end{cases}$$

Let $t_0, \dots, t_n \in \mathbb{N}$ s.t. $t_{i+1} \in \{t_i, t_i + 1\}$

$$\mathbb{P}[\tilde{T}_0 = t_0, \dots, \tilde{T}_n = t_n]$$

$$= \sum_k \mathbb{P}[\bar{\tau} = k, T'_0 = t_0, \dots, T'_k = t_k, T_{k+2} = t_{k+1}, \dots, T_{n+1} = t_n]$$

$$= \sum_k \mathbb{P}[\bar{\tau} = k, T'_0 = t_0, \dots, T'_k = t_k, T_{k+2} - T_{k+1} = t_{k+1} - t_k, \dots, T_{n+1} - T_n = t_n - t_{n-1}]$$

$$= \sum_k \mathbb{P}[\bar{\tau} = k, T'_0 = t_0, \dots, T'_k = t_k] \times \left(\frac{1}{2}\right)^{n-k}$$

$$= \underbrace{\sum_k \mathbb{P}[\bar{\tau} = k | T'_0 = t_0, \dots, T'_n = t_n]}_{= 1} \times \left(\frac{1}{2}\right)^k \times \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2}\right)^n.$$

Proof of the proposition.

Let $(X_n)_{n \geq 0}$ be a SRW from α . Independently, let $(T_n), (\tilde{T}_n)$ as in the Lemma. Set

$$Z_n = X_{T_n} \quad \text{and} \quad \tilde{Z}_n = X_{\tilde{T}_n}$$

$$\mathbb{P}[\exists n_0 \geq 0 \text{ s.t. } \forall n \geq n_0, Z_{n+1} = \tilde{Z}_n]$$

$$\geq \mathbb{P}[\exists n_0 \geq 0 \text{ s.t. } \forall n \geq n_0, T_{n+1} = \tilde{T}_n] = 1 \quad \blacksquare$$

6 PROOF OF THE MAIN THEOREM ($l > 0 \Leftrightarrow \text{no LP}$)

Fix $(Y_n)_{n \geq 0}$ L RW from 0. Define

<ul style="list-style-type: none"> • $\tilde{P} := \lim_{n \rightarrow \infty} \frac{Y_n}{n}$ • $\tilde{h} = \lim_{n \rightarrow \infty} \frac{H(Y_n)}{n}$ • $\tilde{\sigma} = \bigcap_{n \geq 0} \sigma(Y_n, Y_{n+1}, \dots)$
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Exercise : - Prove $\tilde{P} = \frac{l}{2}$

, Express \tilde{h} as a function of h .

Exercise: As in Sections 1 and 2, check that

$$\tilde{P} > 0 \Leftrightarrow \tilde{h} > 0 \Leftrightarrow \tilde{\sigma} \text{ non trivial.}$$

Thm: The following are equivalent.

(i) $\tilde{\mathcal{G}}$ non trivial

(ii) G has no LP.

Corollary: $P > 0 \Leftrightarrow G$ has no LP

Proof of the Thm:

(ii) \Rightarrow (i) Assume $\tilde{\mathcal{G}}$ non trivial.

Let h bounded harmonic. By the martingale convergence theorem, the limit

$$z = \lim_{n \rightarrow \infty} h(Y_n)$$

exists a.s. and in L' , and $E[z] = E[h(Y_0)] = h(0)$.

z is $\tilde{\mathcal{G}}$ measurable, hence $z = h(0)$ a.s.

Now let $x \in V$ and $m \geq |x|$.

$$h(x) = E \left[h(Y_{n+m}) \mid Y_m = x \right] = \underbrace{E \left[h(Y_{m+n}) \mathbb{1}_{Y_m=x} \right]}_{\mathbb{P}[Y_m=x]} \xrightarrow[n \rightarrow \infty]{L'} h(0),$$

hence $h(x) = h(0)$, h is constant.

(i) \Rightarrow (ii) Let $A \in \mathcal{F}$. Let $x \in V$

Claim: $\forall n \geq d(0, x) . \mathbb{P}[A | Y_n = x] = \mathbb{P}[A | Y_{n+1} = x]$

Pf of the claim:

Let $k \geq 0$. $A \in \sigma(Y_{n+k}, Y_{n+k+1}, \dots)$

Hence $1_A = F(Y_{n+k}, Y_{n+k+1}, \dots)$ a.s.

for some measurable $F: V^{\mathbb{N}} \rightarrow \{0, 1\}$

Let $(z_k)_{k \geq 0}, (\tilde{z}_k)_{k \geq 0}$ two coupled LRV from x s.t. $\mathbb{P}[\exists k_0 \geq 1 \forall k \geq k_0 z_k = \tilde{z}_{k-1}] = 1$

$$\mathbb{P}[A | Y_n = x] = \mathbb{E}[F(Y_{n+k}, \dots) | Y_n = x]$$

$$\stackrel{\text{MP}}{=} \mathbb{E}[F(z_k, z_{k+1}, \dots)]$$

$$\mathbb{P}[A | Y_{n+1} = x] = \mathbb{E}[F(Y_{n+k}, \dots) | Y_{n+1} = x]$$

$$\stackrel{\text{MP}}{=} \mathbb{E}[F(\tilde{z}_{k-1}, \tilde{z}_k, \dots)]$$

$$|\mathbb{P}[A | Y_n = x] - \mathbb{P}[A | Y_{n+1} = x]|$$

$$\leq \mathbb{E}|F(z_k, z_{k+1}, \dots) - F(\tilde{z}_{k-1}, \tilde{z}_k, \dots)|$$

$$\leq \mathbb{P}[\forall i \geq k z_i = \tilde{z}_{i-1}] \xrightarrow{k \rightarrow \infty} 0 \quad \blacksquare$$

Define $h(x) = \mathbb{P}[A | Y_n = x]$ (indep. of the choice of $n \geq d(0, x)$)

h is harmonic

$$\begin{aligned}
 h(x) &= \sum_y \underbrace{\mathbb{P}[A | Y_{n+1} = y, Y_n = x]}_{= \mathbb{P}[A | Y_{n+1} = y]} \underbrace{\mathbb{P}[Y_{n+1} = y | Y_n = x]}_{\frac{1}{2} \mathbb{1}_{x=y} + \frac{1}{2d} \mathbb{1}_{x \neq y}} \\
 &= h(y) \\
 &= \frac{1}{2} h(x) + \frac{1}{d} \sum_{y \sim x} h(y)
 \end{aligned}$$

h is non constant

Assume $h = c$ constant.

$$\begin{aligned}
 c &= h(Y_n) = \mathbb{E}[\mathbb{1}_A | Y_n] \\
 &= \mathbb{E}[\mathbb{1}_A | Y_0, \dots, Y_n]
 \end{aligned}$$

$\xrightarrow{n \rightarrow \infty}$ $\mathbb{1}_A$ a.s.
 non constant. (by Levy's 0-1 Law
see [Durrett, probability theory & examples])

contradiction.