Reading group on random nodal lines at ETHZ, Autumn 2019, 1^{st} talk: introduction.

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Contents

1 The objects

1.1 The general framework

We consider a centered Gaussian field indexed by the plane: $(f(x))_{x\in\mathbb{R}^2}$. We assume that its law is invariant under translation, i.e. that there exists a function $\kappa : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbb{E}[f(x)f(y)] = \kappa(x-y).$$

The function κ is called the **covariance function**. We assume that f is a.s. C^∞ (which is equivalent to the fact that κ is C^∞)[1]. We also assume that the law of f is invariant under rotation (i.e. that κ is radial) and that we are not in the degenerate case $f = 0$.

Given a level $\ell \in \mathbb{R}$, we are interested in the geometric properties of the level set $\{f = \ell\}$. We can prove that, under the above hypothesis, for every ℓ a.s. there is no $x \in \mathbb{R}^2$ such that $f(x) = \ell$ and $\bigtriangledown_x f = 0$ (see Remark 1.1). Therefore, a.s. $\{f = \ell\}$ is a union of smooth loops (and maybe of one or several unbounded smooth paths). When $\ell = 0$, a connected component of $\{f = \ell\}$ is called a **nodal line**.

We will be interested in three main properties:

[1] More precisely, this implies that κ is C^∞ and if κ is C^∞ then there exists a modification of f which is C^∞, see for instance Appendix A in [NS16]

1. **The length.** Can we estimate the expectation and the variance of length$(\{f = \ell\} \cap [-R,R]^2)$? Do we have a LLN and a CLT? Remark: this is a **local** quantity: in order to compute the length of $\{f = \ell\}$ in a domain, we can divide the domain in subdomains, compute the length in each subdomain, and then sum.

2. **The number of loops.** What about the number of connected components of $\{f = \ell\} \cap [-R,R]^2)$ (expectation, variance, LLN, CLT)? Remark: this is **not** a local quantity. However, if we know this quantity in some subdomains, then it gives some information on the quantity in the whole domain (for instance we have a subadditivity property).

3. **Percolation properties.** Is there an unbounded level line? What about the event that there is a crossing of $[-R,R]^2$ from left to right included in $\{f = \ell\}$? Remark: this is **not at all** a local property.

Remark 1.1. By using that a.s. convergence implies L^2 convergence for Gaussian variables, we obtain that for any $\alpha, \beta \in \mathbb{N}^2$, $\mathbb{E}[\partial_\alpha f(x) \partial_\beta f(y)] = (-1)^{|\beta|} \partial_\alpha \partial_\beta \kappa(x-y)$. In particular, for every x, $f(x)$ is independent of $\bigtriangledown_x f$. By using the invariance under rotation, one also obtains that the coordinates of $\bigtriangledown_x f$ are independent. As a result, $(f(x), \bigtriangledown_x f)_x$ is a three dimensional non-degenerate smooth Gaussian field indexed by a space of dimension $2 < 3$. Now, the fact that for every $\ell \in \mathbb{R}$ a.s. there exist no point x at which this vector equals $(\ell, 0, 0)$ comes from general theorems on random fields, see for instance Lemma 11.2.10 of [AT07].

In these notes, we will only consider fields such that κ goes to 0 at infinity, so that we can hope to have some - possibly very weak - spatial "quasi-independence" property.

1.2 Examples of Gaussian fields

A. The Bargmann-Fock (B-F) field. The B-F field is defined by the covariance function $\kappa(x) = e^{-|x|^2/2}$ and can be realized as the following real-analytic function:

$$f(x) = f(x_1, x_2) = e^{-|x|^2/2} \sum_{i,j \in \mathbb{N}} a_{i,j} \frac{x_1^i x_2^j}{\sqrt{i!j!}}, \tag{1}$$

where the $a_{i,j}$ are i.i.d. $\mathcal{N}(0,1)$ (the convergence of the series is a.s. uniform on any compact).

The B-F field can also be seen as the local limit of a model of random homogeneous polynomials on the sphere: the **Kostlan ensemble**, which is defined as follows. Let $d \in \mathbb{N}$ (that we will let go to $+\infty$) and define a random homogeneous polynomial of degree d:

$$\forall (x_0, x_1, x_2) \in \mathbb{S}_2,\ P_d(x_0, x_1, x_2) := \sum_{i,j,k \in \mathbb{N},\, i_0+i_1+i_2=d} \sqrt{\frac{d!}{i_0!i_1!i_2!}} x_0^{i_0} x_1^{i_1} x_2^{i_2},$$

where the a_{i_0,i_1,i_2} are i.i.d. $\mathcal{N}(0,1)$. The law of P_d (on the space $\mathbb{R}_d^{hom}[x_0,x_1,x_2]$) is invariant under composition by any orthogonal matrix. This is not the only such law (even up to multiplication by a constant). However, this is the only law on the space $\mathbb{C}_d^{hom}[x_0,x_1,x_2]$ which is invariant under composition by any unitary matrix. If we zoom at scale $1/\sqrt{d}$ then the model converges to the B-F field.

Because of this property, the B-F field is often called the **algebraic model**.

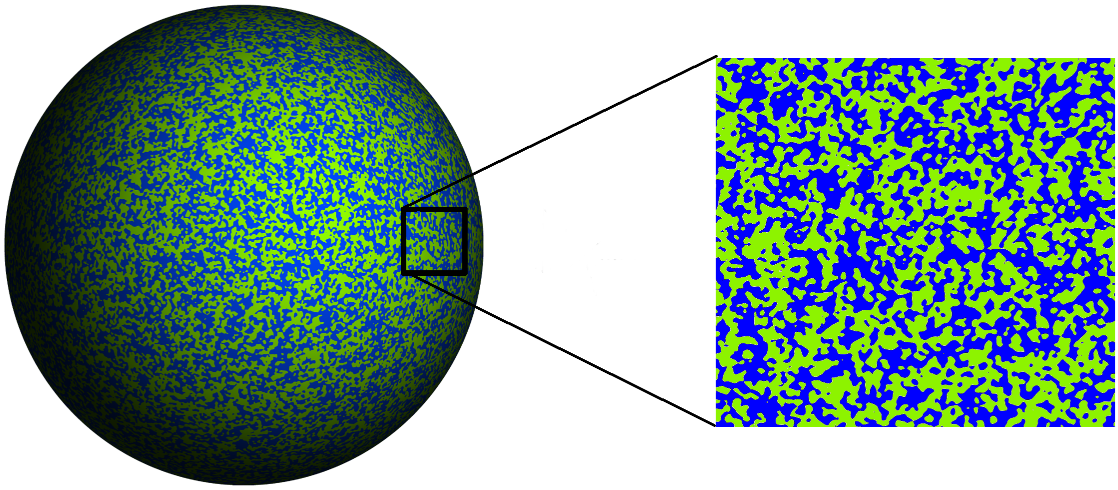

Figure 1: The Kostlan ensemble converges to the B-F field (simulations Vincent Beffara and Alejandro Rivera). In blue: the set where the function is positive; in green: the set where it is negative.

B. The random planar wave (RPW). The RPW model is defined by the covariance function $\kappa(x) = J_0(|x|)$. We recall that that the m^{th} Bessel function J_m is defined by $J_m(r) = \frac{1}{\pi}\int_0^\pi \cos(mu - r\sin(u))du = \sqrt{\frac{2}{\pi r}}\cos(r - m\pi/2 - \pi/4) + O_{r\to+\infty}(1/r)$. The RPW can be realized as the following sum:

$$f(x) = f(re^{i\theta}) = \sum_{m\in\mathbb{N}} (a_m\cos(m\theta) + b_m\sin(m\theta))J_m(r),$$

where the a_m are i.i.d. $\mathcal{N}(0,1)$. One can prove[2] that a.s. $\triangle f = -f$: the RPW is a random eigenfunction of the Laplacian.

The RPW can also be seen as the local limit of a model of random spherical harmonics on the sphere. The Laplacian on the sphere is defined as follows: if $g:\mathbb{S}^2\to\mathbb{R}$ is C^2, we let $\triangle_{\mathbb{S}_2}g = (\triangle_{\mathbb{R}^3}\widetilde{g})_{|\mathbb{S}^2}$ where $\widetilde{g}(x) := g(x/|x|)$. Then, the eigenvalues of $-\triangle_{\mathbb{S}_2}$ are the numbers $k(k+1)$ and are of multiplicity $2k+1$ i.e. the L^2 space of functions $g:\mathbb{S}^2\to\mathbb{R}$ such that $-\triangle_{\mathbb{S}_2} g = k(k+1)g$ is a Hilbert space of dimension $2k+1$. We let h_k be

[2] By using that $\triangle = \partial_{r,r} + 1/r\partial_r + 1/r^2\partial_{\theta,\theta}$ and that $r^2J_m''(r) + rJ_m'(r) + (r^2-m^2)J_m(r) = 0$.

a standard Gaussian variable on this space. Then, if we zoom at scale $1/k$, the model converges to the RPW.

Because of this property, the RPW field is often called the **Riemannian model**.

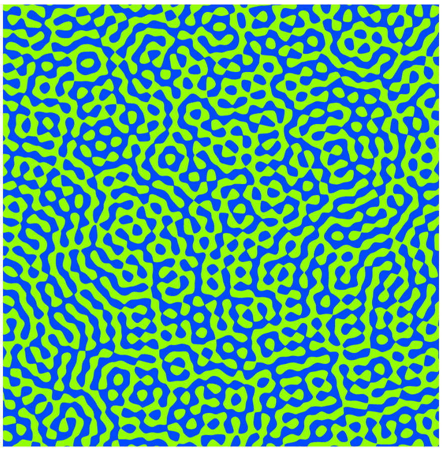

Figure 2: The RPW (simulation by Vincent Beffara).

Remark 1.2. The B-F field and the RPW are **very rigid** (the B-F is a.s. real-analytic and the RPW is a.s. an eigenfunction of the Laplacian). For any open set $U \subset \mathbb{R}^2$, $(f(x))_{x\in\mathbb{R}^2}$ is measurable with respect $(f(x))_{x\in U}$. In particular, we have **no spatial Markov property** here!

C. Fields with polynomial decay of correlation. Below we will also consider fields g_α for every $\alpha > 0$ such that $0 < \kappa(x) = \kappa_\alpha(x) \asymp_\infty |x-y|^{-\alpha}$ (with sometimes further conditions on κ_α).

2 Some results

In this section, we state some results for Gaussian fields with the hypothese of Subsection 1.1. We still use the abbreviations/notations B-F, RPW and g_α.

We write "w.s.w.h." for "with some weak hypotheses" when we are not precise on the exact assumptions we need on the covariance and that these assumptions are very weak - e.g. weak non-degeneracy assumptions or weak assumptions about the speed of decay of the derivatives of κ.

2.1 The length

Let $L_R(\ell) = \text{length}(\{f = \ell\} \cap [-R,R]^2)$. First note that for every ℓ there exists $c > 0$ such that $\mathbb{E}[L_R(\ell)] = cR^2$. Indeed, by dividing the squares in 2×2 squares we have $\mathbb{E}[L_R(\ell)] = R^2\mathbb{E}[L_1(\ell)]$ (we leave as an exercice that $\mathbb{E}[L_1(\ell)]$ is finite and positive). We have the following concentration properties.

Theorem 2.1.
- *Consider the B-F field or a field g_α, $\alpha > 2$ (w.s.w.h.)*[3]*. Then, for every ℓ there exists $c > 0$ such that* $\mathrm{Var}(L_R(\ell)) \sim cR^2$ [KL01] *i.e. $L_R(\ell)$ behaves like a sum of R^2 i.i.d. variables, which is what one could except if one sees the hypothesis $\alpha > 2$ as an hypothesis that implies some independence.*

- *For the random spherical harmonics, for every ℓ there exists $c > 0$ such that* $\mathbb{E}[kL_k(\ell)] \sim ck^2$ *where $L_k(\ell)$ is the length of $\{h_k = \ell\}$* [Bér85]*. (Note that $kL_k(\ell)$ is the analogue of $L_R(\ell)$ with $k \simeq R$.) Moreover, if $\ell \neq 0$ there exists $c' > 0$ such that* $\mathrm{Var}(kL_k(\ell)) \sim c'k^3$ [Ros16]*, and there exists $c'' > 0$ such that* $\mathrm{Var}(kL_k(0)) \sim c''k^2 \log(k)$ [Wig10].

- *For the RPW, if $\ell \neq 0$ it is expected - and partial results are known* [Vid18] *- that there exists $c > 0$ such that* $\mathrm{Var}(L_R(\ell)) \sim cR^3$*, and it is proven that there exists $c' > 0$ such that* $\mathrm{Var}(L_R(0)) \sim c'R^2 \log(R)$ [NPR19].

The above surprising phenomenon at $\ell = 0$ for the RPW and the spherical harmonics is called **Berry's cancellation**.

Note that the above implies a **LLN**: $L_R(\ell)/R^2$ converges a.s. and in L^2 to some positive constant. Actually, the LLN is also a direct consequence of ergodic theorems since it is known that, if κ goes to 0 at infinity, then the field is ergodic (see Theorem 6.5.4 of [Adl10]).

A **CLT** is also known: $\frac{L_R(\ell)-\mathbb{E}[L_R(\ell)]}{\sqrt{\mathrm{Var}(L_R(\ell))}}$ converges in distribution to $\mathcal{N}(0,1)$. More precisely, the CLT is proved (i) for the B-F and for g_α, $\alpha > 2$ (w.s.w.h., [KL01]), (ii) for the random spherical harmonics [Ros16, MRW17], (iii) in the case $\ell = 0$ for the RPW [NPR19]. The CLT is not known for the RPW at $\ell \neq 0$ (as explained above, only partial results for the variance are known in this case).

The "Berry's cancellation" phenomenon at $\ell = 0$ actually implies that, for the RPW, $L_R(0)$ can be approximated by a random variable that belongs to a specific L^2 space (the so-called 4^{th} Wiener chaos space). This makes the study easier (in particular, in this space convergence in distribution to the Gaussian is implied by the convergence of the fourth moment - this is the so-called "Theorem four", see [NP05]). We will not say more about this here and we refer for instance to the survey [Ros18].

2.2 The number of connected components

Let $N_R(\ell)$ denote the number of connected components of $\{f = \ell\} \cap [-R, R]^2$. Contrary to the length which is a "local" quantity, the LLN is not a direct consequence of ergodic theorems: in the case of the number of connected components, one has to deal with "boundary terms". The following LLN is proved:

Theorem 2.2. [NS16] *W.s.w.h., we have a LLN: for every ℓ there exists $c > 0$ such that $N_R(\ell)/R^2$ converges a.s. and in L^1 to c.*

[3]Here the positivity of the covariance of g_α is not important: what is crucial is just that the covariance is L^1.

No CLT is known. What about estimates on the variance? It is conjectured that the variance behaves similarly as in the case of the length. More precisely, it is conjectured that, for any ℓ, the variance is asymptotic to R^2 for the B-F field and for g_α, $\alpha > 2$ (w.s.w.h.).[4] Also, it is conjectured that the variance of $N_R(\ell)$ for the RPW is asymptotic to R^3 if $\ell \neq 0$ but is much less than R^3 if $\ell = 0$. We have the following partial results.

Theorem 2.3. [BMM19]

- *For the B-F field, if* $|\ell| \geq 1.37$ *then* $\mathrm{Var}(N_R(\ell)) \geq c_\ell R^2$. *This also holds for* $|\ell|$ *sufficiently large for* g_α, $\alpha > 2$ *with some strong hypotheses on* κ_α.
- *For the RPW, if* $|\ell| \geq 1$ *then* $\mathrm{Var}(N_R(\ell)) \geq c_\ell R^3$.

Moreover (w.s.w.h.) the following is a corollary of the work [EF16] on critical points of Gaussian fields: $\mathrm{Var}(N_R(\ell)) \leq CR^4$. Let us also point out that a proof that $\mathrm{Var}(N_R(0)) \geq cR^a$ for some $a > 0$ for the RPW has been announced [NS].

2.3 Percolation

The first question we ask in this section is "is there an unbounded component in $\{f = \ell\}$"? It is known that, if $\kappa \geq 0$, then a.s. for every $\ell \in \mathbb{R}$ there is no such unbounded component [Ale96]. Note that (since the level lines are smooth), the existence of an unbounded connected component in $\{f = 0\}$ is equivalent to the equivalent property for $\{f > 0\}$ (and also for $\{f < 0\}$). In particular, the absence of an unbounded component in $\{f = 0\}$ implies the absence of an unbounded component in $\{f = \ell\}$ for every ℓ.

For the RPW (for which we do not have $\kappa \geq 0$!), the fact that $\{f = 0\}$ has no unbounded component is still a conjecture. It is actually expected that under very weak assumptions (and maybe even for any field that satisfies the general hypotheses of these notes), $\{f = 0\}$ has no unbounded component.

The following is a more quantitative version with stronger hypotheses. We let $\mathrm{Arm}_\ell(R)$ denote the event that there is a loop at level ℓ that crosses both the circle of radius 1 and the circle of radius R. Note that $\mathbb{P}[\mathrm{Arm}_R(\ell)]$ converges to the event that there is an unbounded component in $\{f = \ell\}$ that crosses the circle of radius 1.

Theorem 2.4.

- [BG17, BM18, RV17b, MV18] *Consider the B-F field, the field* g_α, $\alpha > 4$ *(w.s.w.a.), or the field* g_α, $\alpha \in]2, 4]$ *with some strong hypotheses on* κ_α. *Then, (i) there exist* $C, c > 0$ *such that* $\mathbb{P}[\mathrm{Arm}_0(R)] \leq CR^{-c}$ *and (ii) for any* $a > 0$, *the event that there is a crossing from left to right of* $[-aR, aR] \times [-R, R]$ *by a nodal line has probability bounded away from* 0 *and* 1 *uniformly in* R.
- [RV17a, MV18, Riv19] *For the B-F and for* g_α, $\alpha > 2$ *with some strong hypotheses on* κ_α, *for every* $\ell \neq 0$ *there exist* $C, c > 0$ *such that* $\mathbb{P}[\mathrm{Arm}_\ell(R)] \leq Ce^{-cR}$.

From a percolation point of view, this might seem surprising to study the level lines, which are the interfaces between the excursion sets $\{f > \ell\}$ and $\{f < \ell\}$, rather than these excursion sets themselves. Let us study these excursion sets.

[4] Here the positivity of κ_α is not important, what is crucial is that $|\kappa_\alpha(x)| \leq C|x|^{-\alpha}$.

For the fields for which the exponential decay of Theorem 2.4 is proved, it is also proved that the percolation critical point is 0 in the sense that there is a.s. an unbounded component in $\{f > \ell\}$ if $\ell < 0$ but this is a.s. not the case if $\ell \geq 0$ (as explained above, the case $\ell \geq 0$ comes from [Ale96]). A finite size version of this property is that the probability that there is a crossing from left to right of $[-aR, aR] \times [-R, R]$ included in $\{f > \ell\}$ goes to 0 (exponentially fast) if $\ell > 0$ and goes to 1 (exponentially fast) if $\ell < 0$. In the previous subsections, we stated concentration results for the length and the number of connected component. Let us keep this point of view: the finite size version of the phase transition is a consequence of the following concentration result.

Theorem 2.5. [RV17a, MV18, Riv19][5] *Fix some $a > 0$. Let T_R denote the threshold function i.e. the supremum of all levels ℓ such that there is a crossing from left to right of $[-aR, aR] \times [-R, R]$ included in $\{f > \ell\}$. Then, for the B-F field and for g_α, $\alpha > 2$ with some strong hypotheses on κ_α, $\mathrm{Var}(T_R)$ goes to 0 as R goes to $+\infty$ polynomially fast.*

We will see later that w.s.w.h. on the field, we have that $\mathrm{Var}(T_R)$ is bounded uniformly in R.

Note that there exists a left-right crossing in $\{f \geq T_R\}$ and that there is a top-bottom crossing in $\{f \leq T_R\}$. W.s.w.h., a.s. for every level ℓ there is at most one critical point at level ℓ. That implies that there exists a.s. a unique point S_R (the "saddle point") such that: (i) $f(S_R) = T_R$, (ii) there exists a left-right crossing $\gamma \subset \{f \geq T_R\}$ such that $S_R \in \gamma$ and $f > T_R$ on $\gamma \setminus \{S_R\}$ and (iii) there exists a top-bottom crossing $\gamma \subset \{f \leq T_R\}$ such that $S_R \in \gamma$ and $f < T_R$ on $\gamma \setminus \{S_R\}$. We will see that the fact that $\mathrm{Var}(T_R)$ goes to 0 is equivalent to some "spatial chaotic behavior" of the saddle point.

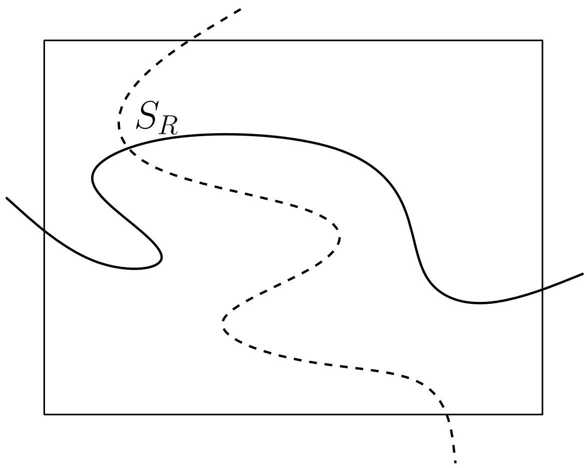

Figure 3: The saddle point of the crossing event. In full line: $f > T_n$; in dashed line: $f < T_n$; $f(S_n) = T_n$.

[5] The concentration point of view comes from [Riv19].

One of the main questions in the study of percolation of Gaussian fields is: do the models behave like Bernoulli percolation? It is actually conjectured that for the B-F field, the RPW and for g_α, $\alpha > 3/2$, the model has the same scaling limit as Bernoulli percolation (i.e. the probabilities of crossings at level 0 are given by the Cardy formula and the scaling limit of the interface is the SLE$_6$ process, ...).[6] In particular, it is believed that for these processes we have $\mathrm{Var}(T_R) = R^{-3/2+o(1)}$. As we will see later, this property is not true for g_α if $\alpha \in]0, 3/2[$ (here the positivity of the covariance of g_α is crucial).

It might not be very surprising that we expect that the model has the same scaling limit as Bernoulli percolation when the covariance decays fast since this sems to be equivalent to some spatial independence property (we will say more about this below). What might be more surprinsing is that it is believed that the percolation model constructed by using the RPW - whose covariance decays like $1/\sqrt{|x-y|}$ - behaves like Bernoulli percolation and that this is not the case for $g_{1/2}$. The idea is that the oscillations of the covariance of the RPW should help to have some spatial mixing.

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