Reading group on random nodal lines at ETHZ and Zürich University, Autumn 2019, $1^{st}$ and $2^{nd}$ talks.

# Overview and concentration results for nodal lines.

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## Contents



## 1 The objects

### 1.1 The general framework

We consider a centered Gaussian field indexed by the plane: $(f(x))_{x\in\mathbb{R}^2}$. We assume that its law is invariant under translations, i.e. that there exists a function $\kappa : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbb{E}[f(x)f(y)] = \kappa(x-y).$$

The function $\kappa$ is called the **covariance function**. We assume that $f$ is a.s. $C^\infty$ (which is equivalent to the fact that $\kappa$ is $C^\infty$)[1]. We also assume that the law of $f$ is invariant under rotation (i.e. that $\kappa$ is radial) and that we are not in the degenerate case $f = 0$.

Given a level $\ell \in \mathbb{R}$, we are interested in the geometric properties of the level set $\{f = \ell\}$. We can prove that, under the above hypotheses, for every $\ell$ a.s. there is no $x \in \mathbb{R}^2$ such that $f(x) = \ell$ and $\bigtriangledown_x f = 0$ (see Remark 1.1). Therefore, a.s. $\{f = \ell\}$ is a union of smooth loops (and maybe of one or several unbounded smooth paths). When $\ell = 0$, a connected component of $\{f = \ell\}$ is called a **nodal line**.

We will be interested in three following quantities/properties:

1. **The length.** Can we estimate the expectation and the variance of the length of $\{f = \ell\} \cap [-R, R]^2$? Do we have a LLN and a CLT? Remark: this is a **local** quantity: in order to compute the length of $\{f = \ell\}$ in a domain, we can divide the domain in subdomains, compute the length in each subdomain, and then sum.

2. **The number of loops.** What about the number of connected components of $\{f = \ell\} \cap [-R, R]^2$ (expectation, variance, LLN, CLT)? Remark: this is **not** a local quantity. However, if we know this quantity in some subdomains, then it gives some information on the quantity in the whole domain (for instance we have a subadditivity property).

3. **Percolation properties.** Is there an unbounded level line? What about the event that there is a crossing of $[-R, R]^2$ from left to right included in $\{f = \ell\}$? Remark: this is **not at all** a local property.

**Remark 1.1.** By using that a.s. convergence implies $L^2$ convergence for Gaussian variables, we obtain that for any $\alpha, \beta \in \mathbb{N}^2$, $\mathbb{E}[\partial_\alpha f(x) \partial_\beta f(y)] = (-1)^{|\beta|} \partial_\alpha \partial_\beta \kappa(x - y)$. In particular, for every $x$, $f(x)$ is independent of $\bigtriangledown_x f$. By using the invariance under rotation, one also obtains that the coordinates of $\bigtriangledown_x f$ are independent. As a result, $(f(x), \bigtriangledown_x f)_x$ is a three dimensional non-degenerate smooth Gaussian field indexed by a space of dimension $2 < 3$. Now, the fact that for every $\ell \in \mathbb{R}$ a.s. there exist no point $x$ at which this vector equals $(\ell, 0, 0)$ comes from general theorems on random fields, see for instance Lemma 11.2.10 of [AT07].

In these notes, we also assume that $\kappa$ goes to 0 at infinity, so that we can hope to have some - possibly very weak - spatial "quasi-independence" property.

### 1.2 Examples of Gaussian fields

**A. The Bargmann-Fock (B-F) field.** The B-F field is defined by the covariance function $\kappa(x) = e^{-|x|^2/2}$ and can be realized as the following real-analytic function:

$$f(x) = f(x_1, x_2) = e^{-|x|^2/2} \sum_{i,j \in \mathbb{N}} a_{i,j} \frac{x_1^i x_2^j}{\sqrt{i! j!}}, \tag{1}$$

[1] More precisely, this implies that $\kappa$ is $C^\infty$ and if $\kappa$ is $C^\infty$ then there exists a modification of $f$ which is $C^\infty$, see for instance Appendix A in [NS16].

where the $a_{i,j}$ are i.i.d. $\mathcal{N}(0,1)$ (the convergence of the series is a.s. uniform on any compact).

The B-F field can also be seen as the local limit of a model of random homogeneous polynomials on the sphere: the **Kostlan ensemble**, which is defined as follows. Let $d \in \mathbb{N}$ (that we will let go to $+\infty$) and define a random homogeneous polynomial of degree $d$:

$$\forall (x_0,x_1,x_2) \in \mathbb{S}_2,\ P_d(x_0,x_1,x_2) := \sum_{i,j,k\in\mathbb{N},\, i_0+i_1+i_2=d} \sqrt{\frac{d!}{i_0!i_1!i_2!}} x_0^{i_0} x_1^{i_1} x_2^{i_2},$$

where the $a_{i_0,i_1,i_2}$ are i.i.d. $\mathcal{N}(0,1)$. The law of $P_d$ (on the space $\mathbb{R}_d^{hom}[x_0,x_1,x_2]$) is invariant under composition by any orthogonal matrix. This is not the only such law (even up to multiplication by a constant). However, this is the only law on the space $\mathbb{C}_d^{hom}[x_0,x_1,x_2]$ which is invariant under composition by any unitary matrix. If we zoom at scale $1/\sqrt{d}$ then the model converges to the B-F field:

$$\begin{aligned}
&P_d\left(\sqrt{1-\frac{x_1^2+x_2^2}{d}}, \frac{x_1}{\sqrt{d}}, \frac{x_2}{\sqrt{d}}\right) \\
&= \sum_{i_1,i_2\in\mathbb{N},\, i_1+i_2\le d} a_{d-(i_1+i_2),i_1,i_2} \sqrt{\frac{d!}{(d-(i_1+i_2))!i_1!i_2!}} \left(\sqrt{1-\frac{x_1^2+x_2^2}{d}}\right)^{d-(i_1+i_2)} \frac{x_1^{i_1}x_2^{i_2}}{\sqrt{d}^{i_1+i_2}} \\
&= \sum_{i_1,i_2\in\mathbb{N},\, i_1+i_2\le d} \left(\sqrt{1-\frac{x_1^2+x_2^2}{d}}\right)^{d-(i_1+i_2)} a_{d-(i_1+i_2),i_1,i_2} \frac{x_1^{i_1}x_2^{i_2}}{\sqrt{i_1!i_2!}} \sqrt{\frac{d!}{(d-(i_1+i_2))!d^{i_1+i_2}}}.
\end{aligned}$$

(For any fixed $i_1, i_2$, $\left(\sqrt{1-\frac{x_1^2+x_2^2}{d}}\right)^{d-(i_1+i_2)}$ converges to $e^{-|x|^2/2}$ and $\sqrt{\frac{d!}{(d-i_1+i_2)!d^{i_1+i_2}}}$ converges to 1, and one can deduce[2] from this that the above converges a.s. on every compact to an entire series of the form (1).)

Because of this property, the B-F field is often called the **algebraic model**.

**B. The random planar wave (RPW).** The RPW model is defined by the covariance function $\kappa(x) = J_0(|x|)$. We recall that that the $m^{th}$ Bessel function $J_m$ is defined by $J_m(r) = \frac{1}{\pi}\int_0^\pi \cos(mu - r\sin(u))du = \sqrt{\frac{2}{\pi r}}\cos(r - m\pi/2 - \pi/4) + O_{r\to+\infty}(1/r)$. The RPW can be realized as the following sum:

$$f(x) = f(re^{i\theta}) = \sum_{m\in\mathbb{N}} (a_m\cos(m\theta) + b_m\sin(m\theta))J_m(r),$$

where the $a_m$ are i.i.d. $\mathcal{N}(0,1)$. One can prove[3] that a.s. $\triangle f = -f$: the RPW is a random eigenfunction of the Laplacian.

[2]By being a little quantitative and by choosing that $a_{d-(i_1+i_2),i_1,i_2}$ does not depend on $d \ge i_1+i_2$.
[3]By using that $\triangle = \partial_{r,r} + 1/r\partial_r + 1/r^2\partial_{\theta,\theta}$ and that $r^2J_m''(r) + rJ_m'(r) + (r^2-m^2)J_m(r) = 0$.

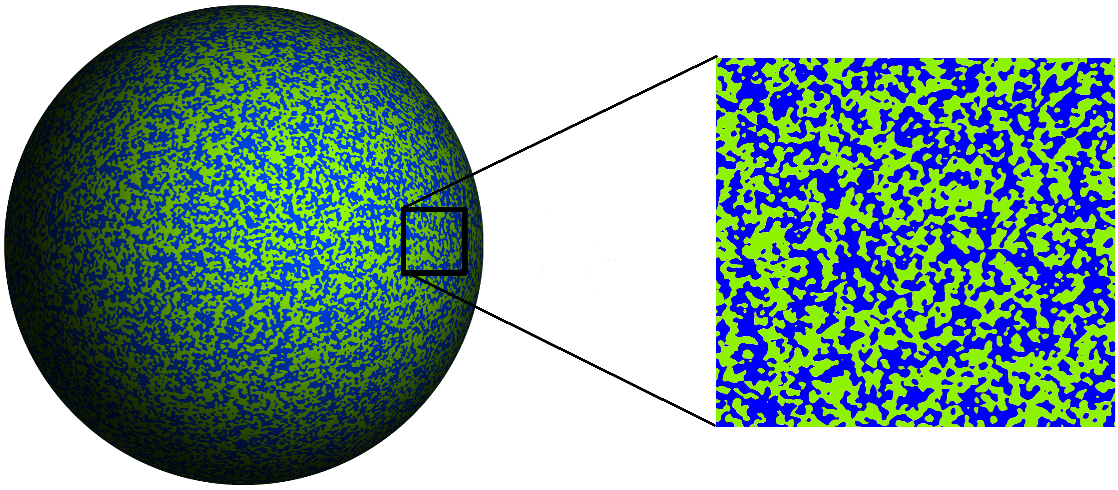

Figure 1: The Kostlan ensemble converges to the B-F field (simulations Vincent Beffara and Alejandro Rivera). In blue: the set where the function is positive; in green: the set where it is negative.

The RPW can also be seen as the local limit of a model of random spherical harmonics on the sphere. The Laplacian on the sphere is defined as follows: if $g : \mathbb{S}^2 \to \mathbb{R}$ is $C^2$, we let $\triangle_{\mathbb{S}_2} g = (\triangle_{\mathbb{R}^3} \widetilde{g})_{|\mathbb{S}^2}$ where $\widetilde{g}(x) := g(x/|x|)$. Then, the eigenvalues of $-\triangle_{\mathbb{S}_2}$ are the numbers $k(k+1)$ and are of multiplicity $2k+1$ i.e. the $L^2$ space of functions $g : \mathbb{S}^2 \to \mathbb{R}$ such that $-\triangle_{\mathbb{S}_2} g = k(k+1)g$ is a Hilbert space $H_k$ of dimension $2k+1$. We let $h_k$ be a standard Gaussian variable on $H_k$ times $(2k+1)^{-1/2}$. These random eigenfunctions of $\triangle_{\mathbb{S}_2}$ are called **random spherical harmonics**. Then, if we zoom at scale $1/k$, the model converges to the RPW.

Because of this property, the RPW field is often called the **Riemannian model**.

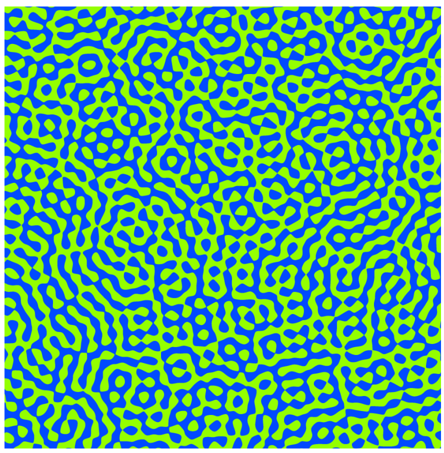

Figure 2: The RPW (simulation by Vincent Beffara).

**Remark 1.2.** The B-F field and the RPW are **very rigid** (the B-F is a.s. real-analytic and the RPW is a.s. an eigenfunction of the Laplacian). For any open set $U \subset \mathbb{R}^2$, $(f(x))_{x\in\mathbb{R}^2}$ is measurable with respect $(f(x))_{x\in U}$. In particular, we have **no spatial Markov property** here!

**C. Fields with polynomial decay of correlation.** Below we will also consider fields $g_\alpha$ for every $\alpha > 0$ such that $0 < \kappa(x) = \kappa_\alpha(x) \asymp_\infty |x-y|^{-\alpha}$ (with sometimes further conditions on $\kappa_\alpha$).

## 2 Some results

In this section, we state some results for Gaussian fields with the hypothese of Subsection 1.1. We try to focus essentially on **concentration results**. We still use the abbreviations/notations B-F, RPW and $g_\alpha$.

We write "w.s.w.h." for "with some weak hypotheses" when we are not precise on the exact assumptions we need on the covariance and that these assumptions are very weak - e.g. weak non-degeneracy assumptions or weak assumptions about the speed of decay of the derivatives of $\kappa$.

### 2.1 The length

Let $L_R(\ell) = \mathrm{length}(\{f=\ell\} \cap [-R,R]^2)$. First note that for every $\ell$ there exists $c > 0$ such that $\mathbb{E}[L_R(\ell)] = cR^2$. Indeed, by dividing the squares in $2\times 2$ squares we have $\mathbb{E}[L_R(\ell)] = R^2\mathbb{E}[L_1(\ell)]$ (we leave as an exercice that $\mathbb{E}[L_1(\ell)]$ is finite and positive). We have the following concentration properties.

**Theorem 2.1.**

- *Consider the B-F field or a field $g_\alpha$, $\alpha > 2$ (w.s.w.h.)*[4]*. Then, for every $\ell$ there exists $c > 0$ such that* $\mathrm{Var}(L_R(\ell)) \sim cR^2$ [KL01] *i.e. $L_R(\ell)$ behaves like a sum of $R^2$ i.i.d. variables, which is what one could except if one sees the hypothesis $\alpha > 2$ as an hypothesis that implies some spatial independence properties.*

- *The analogous result holds for the Kostlan ensemble: For every $\ell$ there exist $c, c' > 0$ such that* $\mathbb{E}[\sqrt{d}L_d(\ell)] \sim cd$ [Kos93] *and* $\mathrm{Var}(\sqrt{d}L_d(\ell) \sim c'd$ [Let19] *where $L_d(\ell)$ is the length of $\{P_d = \ell\}$. (Note that $\sqrt{d}L_d(\ell)$ is the analogue of $L_R(\ell)$ with $d \simeq R^2$.)*

- *For the RPW, if $\ell \neq 0$ it is expected - and partial results are known* [Vid18] *- that there exists $c > 0$ such that* $\mathrm{Var}(L_R(\ell)) \sim cR^3$*, and it is proven that there exists $c' > 0$ such that* $\mathrm{Var}(L_R(0)) \sim c'R^2\log(R)$ [NPR19].

- *The analogous result holds for random spherical harmonics: For every $\ell$ there exists $c > 0$ such that* $\mathbb{E}[kL_k(\ell)] \sim ck^2$ *where $L_k(\ell)$ is the length of $\{h_k = \ell\}$* [Bér85].

[4] Here the positivity of the covariance of $g_\alpha$ is not important: what is crucial is just that the covariance is $L^1$.

*(Note that $kL_k(\ell)$ is the analogue of $L_R(\ell)$ with $k \simeq R$.) Moreover, if $\ell \neq 0$ there exists $c' > 0$ such that* $\mathrm{Var}(kL_k(\ell)) \sim c'k^3$ [Ros16]*, and there exists $c'' > 0$ such that* $\mathrm{Var}(kL_k(0)) \sim c''k^2 \log(k)$ [Wig10].

The above surprising phenomenon at $\ell = 0$ for the RPW and the spherical harmonics is called **Berry's cancellation** [Ber02].

Note that the above implies a **LLN**: $L_R(\ell)/R^2$ converges a.s. and in $L^2$ to some positive constant. Actually, the LLN is also a direct consequence of ergodic theorems since it is known that, if $\kappa$ goes to 0 at infinity, then the field is ergodic (see Theorem 6.5.4 of [Adl10]).

A **CLT** is also known: $\frac{L_R(\ell)-\mathbb{E}[L_R(\ell)]}{\sqrt{\mathrm{Var}(L_R(\ell))}}$ converges in distribution to $\mathcal{N}(0,1)$ [KL01, Ros16, MRW17, NPR19] (let us note that the CLT is not known for the RPW at $\ell \neq 0$).

The "Berry's cancellation" phenomenon at $\ell = 0$ actually implies that, for the RPW, $L_R(0)$ can be approximated by a random variable that belongs to a specific $L^2$ space (the so-called $4^{th}$ Wiener chaos space). This makes the study easier (in particular, in this space convergence in distribution to the Gaussian is implied by the convergence of the fourth moment - this is the so-called "Theorem four", see [NP05]). We will not say more about this here and we refer for instance to the survey [Ros18].

Analogous results are also proven for random harmonics on the torus. In this case, there is no universal behavior: the behavior of the variance highly depends on the subsequence of eigenvalues [KW17, KW17, MPRW16]. Results about the length restricted to some subdomains and about expectation/variance/CLT for other local quantities (such as the number of critical points or the Euler characteristic) are proven in [EF16, EL16, CMW16, CW17, CM18, Tod18, PV19]. See [Zel09, Let16, Let19, LP17, AP18][5] for results about the expectation and the variance of local quantities in much more general settings (and in larger dimensions).

**Some toy models.** Let us study these questions on discrete percolation models. Let us first consider a rhombus made of $n^2$ regular hexagons and let $\mathbb{T}_n$ denote the subset of the regular triangular lattice that correspond to this rhombus (i.e. we put a vertex at the center of each hexagon). Let $(X_v)_{v \in \mathbb{T}_n}$ be i.i.d. $\mathcal{N}(0,1)$, let $\ell \in \mathbb{R}$, and let us color the hexagon in black (resp. white) if $X_v > \ell$ (resp. $X_v < \ell$). The level set is defined as the interface between black and white. As a result, the length of the level set is

$$L_n(\ell) = \sum_{\{v,w\} \text{ edge}} 1_{(X_v-\ell)(X_w-\ell)<0}.$$

Then $\mathbb{E}[L_n(\ell)] \sim c_\ell n^2$ for some $c_\ell > 0$ and

$$\mathrm{Var}(L_n(\ell)) = \sum_{\{v,w\},\{v',w'\}} \mathrm{Cov}(1_{(X_v-\ell)(X_w-\ell)<0}, 1_{(X_{v'}-\ell)(X_{w'}-\ell)<0}).$$

[5] [AP18] contains a self-contained introduction to Malliavin calculus which is an important tools in this area.

The covariance term above is 0 if the edges do not share a vertex, is some strictly positive constant if $\{v, w\} = \{v', w'\}$, and is some non-negative constant (positive if and only if $\ell \neq 0$) if the edges share exactly one vertex. This implies that $\mathrm{Var}(L_n(\ell)) \sim c'_\ell n^2$ for some $c'_\ell > 0$.

Let us study a second toy model which should mimic some behaviours of the RPW. We still consider the $n \times n$ rhombus and some $\ell \in \mathbb{R}$. We also consider $X_1, \cdots, X_n$ i.i.d. $\mathcal{N}(0, 1)$ and we color in black (resp. in white) the $i^{th}$ column of the rombus if $X_i > \ell$ (resp. $X_i < \ell$). This time we have

$$L_n(\ell) = n \sum_{i=1}^{n-1} 1_{(X_i-\ell)(X_{i+1}-\ell)<0}.$$

We have the same behavior as the RPW at non-zero levels: $\mathbb{E}[L_n(\ell)] \sim c_\ell n^2$ and $\mathrm{Var}(L_n(\ell)) \sim c'_\ell n^3$.

We also have a CLT for these two toy models. At $\ell = 0$, it suffices to apply the classical CLT since we deal with sums of i.i.d. (bounded) random variables. In the case $\ell \neq 0$, the variables are only 2-dependent. One can apply a CLT for $m$-dependent variables (for this theorem, a $(2 + \delta)$-moment is needed rather than just a 2nd moment, but the variables that we consider are even bounded).

## 2.2 The number of connected components

Let $N_R(\ell)$ denote the number of connected components of $\{f = \ell\} \cap [-R, R]^2$. Contrary to the length which is a “local” quantity, the LLN is not a direct consequence of ergodic theorems: in the case of the number of connected components, one has to deal with “boundary terms”. The following LLN is proved:

**Theorem 2.2.** [NS16] *W.s.w.h., we have a LLN: for every $\ell$ there exists $c > 0$ such that $N_R(\ell)/R^2$ converges a.s. and in $L^1$ to $c$.*

Asymptotic estimates on the expectation of $N_R(\ell)$ in a much more general setting (and in larger dimensions) are proven in [GW12, GW14b, GW14c, GW14a]. No CLT is known. Some concentration results are known but there are only partial results. We start with a large deviations result.

**Theorem 2.3.** [NS09] *For the spherical harmonics $h_k$, there exists $c > 0$ such that, for every $\varepsilon > 0$ there exist $C', c' > 0$ such that*

$$\mathbb{P}\left[\left|\frac{N_k(0)}{k^2} - c\right| \geq \varepsilon\right] \leq C' e^{-c' k},$$

*where $N_k(0)$ is the number of connected components of $\{h_k = 0\}$.*

For the B-F field and for the fields $g_\alpha$, $\alpha > 4$ (w.s.w.h.),[6] concentration results from below (i.e. upper bounds on $\mathbb{P}\left[\frac{N_R(\ell)}{R^2} - c \leq \varepsilon\right]$) are proved in [RV17b].

What about estimates on the variance? It is conjectured that the variance behaves similarly as in the case of the length. More precisely, it is conjectured that, for any $\ell$, the variance is asymptotic to $R^2$ for the B-F field and for $g_\alpha$, $\alpha > 2$ (w.s.w.h.).[7] Also, it is conjectured that the variance of $N_R(\ell)$ for the RPW is asymptotic to $R^3$ if $\ell \neq 0$ but is much less than $R^3$ if $\ell = 0$. We have the following partial results.

**Theorem 2.4.** [BMM19]

- *For the B-F field, if $|\ell| \geq 1.37$ then $\mathrm{Var}(N_R(\ell)) \geq c_\ell R^2$. This also holds for $|\ell|$ sufficiently large for $g_\alpha$, $\alpha > 2$ with some strong hypotheses on $\kappa_\alpha$.*
- *For the RPW, if $|\ell| \geq 1$ then $\mathrm{Var}(N_R(\ell)) \geq c_\ell R^3$.*

Moreover (w.s.w.h.) the following is a corollary of the work [EF16] on critical points of Gaussian fields: $\mathrm{Var}(N_R(\ell)) \leq CR^4$. Let us also point out that a proof that $\mathrm{Var}(N_R(0)) \geq cR^a$ for some $a > 0$ for the RPW has been announced [NS].

### 2.3 Percolation

The first question we ask in this subsection is "is there an unbounded component in $\{f = \ell\}$"? It is known that, if $\kappa \geq 0$, then a.s. for every $\ell \in \mathbb{R}$ there is no such unbounded component [Ale96]. Note that (since the level lines are smooth), the existence of an unbounded connected component in $\{f = 0\}$ is equivalent to the equivalent property for $\{f > 0\}$ (and also for $\{f < 0\}$). In particular, the absence of an unbounded component in $\{f = 0\}$ implies the absence of an unbounded component in $\{f = \ell\}$ for every $\ell$.

For the RPW (for which we do not have $\kappa \geq 0$!), the fact that $\{f = 0\}$ has no unbounded component is still a conjecture. It is actually expected that under very weak assumptions (and maybe even for any field that satisfies the general hypotheses of these notes), $\{f = 0\}$ has no unbounded component.

The following is a more quantitative version with stronger hypotheses. We let $\mathrm{Arm}_\ell(R)$ denote the event that there is a component of $\{f = \ell\}$ that crosses both the circle of radius 1 and the circle of radius $R$. Note that $\mathbb{P}[\mathrm{Arm}_R(\ell)]$ converges to the event that there is an unbounded component in $\{f = \ell\}$ that crosses the circle of radius 1.

**Theorem 2.5.**

- [BG17], extended in [BM18, RV17b, MV18] *Consider the B-F field, the field $g_\alpha$, $\alpha > 4$ (w.s.w.a.), or the field $g_\alpha$, $\alpha \in ]2, 4]$ with some strong hypotheses on $\kappa_\alpha$. Then, (i) there exist $C, c > 0$ such that $\mathbb{P}[\mathrm{Arm}_0(R)] \leq CR^{-c}$ and (ii) for any $a > 0$, the event that there is a crossing from left to right of $[-aR, aR] \times [-R, R]$ by a nodal line has probability bounded away from 0 and 1 uniformly in $R$.*

[6] Here the positivity of the covariance of $g_\alpha$ is not important: what is crucial is just that $|\kappa_\alpha(x)| \leq |x - y|^{-\alpha}$.

[7] See the footnote 6.

- [RV17a, MV18, Riv19] *For the B-F and for $g_\alpha$, $\alpha > 2$ with some strong hypotheses on $\kappa_\alpha$, for every $\ell \neq 0$ there exist $C, c > 0$ such that $\mathbb{P}[\mathrm{Arm}_\ell(R)] \leq Ce^{-cR}$.*

From a percolation point of view, this might seem surprising to study the level lines, which are the interfaces between the excursion sets $\{f > \ell\}$ and $\{f < \ell\}$, rather than these excursion sets themselves. Let us state some properties of these excursion sets.

For the fields for which the exponential decay of Theorem 2.5 is proved, it is also proved that the percolation critical point is 0 in the sense that there is a.s. an unbounded component in $\{f > \ell\}$ if $\ell < 0$ but this is a.s. not the case if $\ell \geq 0$ (as explained above, the case $\ell \geq 0$ comes from [Ale96]). A finite size version of this property is that the probability that there is a crossing from left to right of $[-aR, aR] \times [-R, R]$ included in $\{f > \ell\}$ goes to 0 (exponentially fast) if $\ell > 0$ and goes to 1 (exponentially fast) if $\ell < 0$. In the previous subsections, we stated concentration results for the length and the number of connected component. Let us keep this point of view: the finite size version of the phase transition is a consequence of the following concentration result.

**Theorem 2.6.** [RV17a, MV18, Riv19, GV19][8] *Fix some $a > 0$. Let $T_R$ denote the threshold function i.e. the supremum of all levels $\ell$ such that there is a crossing from left to right of $[-aR, aR] \times [-R, R]$ included in $\{f > \ell\}$. Then, for the B-F field and for $g_\alpha$, $\alpha > 2$ with some strong hypotheses on $\kappa_\alpha$, $\mathrm{Var}(T_R)$ goes to 0 as $R$ goes to $+\infty$ polynomially fast.*

We will see later that w.s.w.h. on the field, we have that $\mathrm{Var}(T_R)$ is bounded uniformly in $R$.

Note that there exists a left-right crossing in $\{f \geq T_R\}$ and that there is a top-bottom crossing in $\{f \leq T_R\}$. W.s.w.h., a.s. for every level $\ell$ there is at most one critical point at level $\ell$. That implies that there exists a.s. a unique point $S_R$ (the "saddle point") such that: (i) $f(S_R) = T_R$, (ii) there exists a left-right crossing $\gamma \subset \{f \geq T_R\}$ such that $S_R \in \gamma$ and $f > T_R$ on $\gamma \setminus \{S_R\}$ and (iii) there exists a top-bottom crossing $\gamma \subset \{f \leq T_R\}$ such that $S_R \in \gamma$ and $f < T_R$ on $\gamma \setminus \{S_R\}$.

We will see in Section 3 that the fact that $\mathrm{Var}(T_R)$ goes to 0 is equivalent to some "spatial chaotic behavior" of the saddle point.

One of the main questions in the study of percolation of Gaussian fields is: do the models behave like Bernoulli percolation? It is actually conjectured that for the B-F field, the RPW and for $g_\alpha$, $\alpha > 3/2$, the model has the same scaling limit as Bernoulli percolation (i.e. the probabilities of crossings at level 0 are given by the Cardy formula and the scaling limit of the interface is the $\mathrm{SLE}_6$ process, ...).[9] In particular, it is believed that for these processes we have $\mathrm{Var}(T_R) = R^{-3/2+o(1)}$. As we will see later, this property is not true for $g_\alpha$ if $\alpha \in ]0, 3/2[$ (here the positivity of the covariance of $g_\alpha$ is crucial).

It might not be very surprising that we expect that the model has the same scaling limit as Bernoulli percolation when the covariance decays fast since this sems to be equivalent to some spatial independence property (we will say more about this below). What

[8] The concentration point of view comes from [Riv19].

[9] See for instance the lecture notes [Wer07] for a survey on planar Bernoulli percolation.

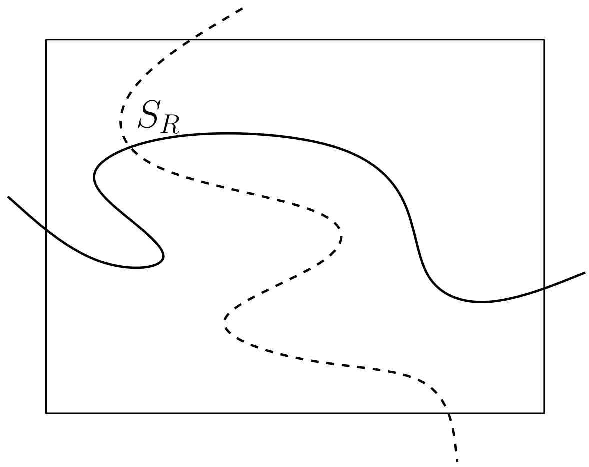


Figure 3: The saddle point of the crossing event. In full line: $f > T_n$; in dashed line: $f < T_n$; $f(S_n) = T_n$.

might be more surprinsing is that it is believed that the percolation model constructed by using the RPW - whose covariance decays like $1/\sqrt{|x-y|}$ - behaves like Bernoulli percolation and that this is not the case for $g_{1/2}$. The idea is that the oscillations of the covariance of the RPW should help to have some spatial mixing.

As we mentioned above, the B-F field and the RPW are very rigid. Concerning the B-F field: (i) on the one hand this is an analytic function so knowing $f$ in a small ball freezes the whole field, (ii) on the other hand the covariance of $f$ decays very fast to 0 at infinity, which suggests a lot of spatial independence. One of the goals is to find suitable classes of events that satisfy spatial independence properties. We have the following.

**Theorem 2.7.** *Let $B_R, B'_R$ be two boxes of size $R$ at distance $R$ from each other and fix some level $\ell$.*

- *For the B-F and for $g_\alpha$, $\alpha > 4$ (w.s.w.h.)*[10], $\mathrm{Cov}(1_{A_R}, 1_{A'_R})$ *goes to* 0 *as $R$ goes $+\infty$ if $A_R$ (resp. $A'_R$) are events measurable with the crossing of a rectangle included in $B_R$ (resp. $B'_R$) or with the number of loops in $B_R$ (resp. $B'_R$).* [RV17b, BMR18]
- *For the B-F and for $g_\alpha$, $\alpha > 2$ with some strong hypothesis on $\kappa_\alpha$, the above is true for any monotonic events.* [MV18]

As a result, the analytic rigidity of the B-F field does not really affect the percolation events and the events measurable with respect to the number of connected components.

It is believed that, for the RPW, the above is true for for percolation events.

[10] See the footnote 6.

## 3 Concentration formulas, superconcentration and chaos

Most of the results that were stated above are estimates on the variance of some quantities related to Gaussian fields i.e. **concentration** results for these fields. In this section, we compute a general formula for the variance of Gaussian functional and we try to apply it to these quantities. This approach is inspired by [Pit12, Cha08]. In particular, the ideas from [Cha08] will help us to: (i) derive some "spatial chaos property", (ii) to prove that if the covariance is positive and decays much faster than $|x|^{-3/2}$, then the associated percolation model does not behave like Bernoulli percolation and (iii) prove some estimates on the variance of the number of connected components for analogous discrete Gaussian models.

### 3.1 A formula for the variance

This subsection is based on [Cha08]. We prove a formula for the variance of a functional of a Gaussian vector.

**Proposition 3.1.** *Let $K$ be an $n \times n$ covariance matrix and let $X, \tilde{X}$ be two independent centered Gaussian vectors of covariance $K$. For each $t \geq 0$, let $X_t = e^{-t}X + \sqrt{1-e^{-2t}}\tilde{X}$. Then, for each $L^2$ continuous function $F : \mathbb{R}^n \to \mathbb{R}$ that satisfies that a.s. for every $i$, $\partial_i F$ exists and is in $L^\infty$, we have*

$$\mathrm{Var}(F(X)) = \int_0^{+\infty} e^{-t} \sum_{1 \leq i,j \leq n} K(i,j)\mathbb{E}[\partial_i F(X) \partial_j F(X_t)]dt.$$

*Moreover, the quantity $\sum_{1 \leq i,j \leq n} K(i,j)\mathbb{E}[\partial_i F(X)\partial_j F(X_t)]$ is non-negative and non-increasing in $t$.*

*Proof.* Let us first deal with the i.i.d. case $K = Id_n$. We have

$$\mathrm{Var}(F(X)) = \mathbb{E}\left[F(X)(F(X) - F(X'))\right] = \int_0^\infty \frac{d}{dt}\mathbb{E}\left[F(X)F(X_t)\right]dt.$$

One can prove that

$$\frac{d}{dt}\mathbb{E}\left[F(X)F(X_t)\right] = e^{-t}\sum_i \mathbb{E}\left[\partial_i F(X)\partial_i F(X_t)\right],$$

which gives the formula in the i.i.d. case. In order to obtain the formula in the general case, one can consider $Y, \tilde{Y}$ independent Gaussian vectors of covariance $Id_n$, let $X = \sqrt{K}Y$, $\tilde{X} = \sqrt{K}\tilde{Y}$, and apply the formula in the i.i.d. case to the function $G : x \mapsto G(\sqrt{K}x)$. One thus obtains the result since

$$\sum_i \mathbb{E}\left[\partial_i G(Y)\partial_i G(Y_t)\right] = \sum_{i,j} K(i,j)\mathbb{E}\left[\partial_i F(X)\partial_j F(X_t)\right].$$

Let us now prove the non-negativity and the monotonicity properties. By the above, it is sufficient to prove this in the i.i.d. case. So let us assume that $K = Id_n$. Next, we note that $P_t : F \mapsto \mathbb{E}[F(X_t) \mid X_0 = \bullet]$ is a symmetric Markov semi-group (this is the so-called Ornstein-Uhlenbeck semi-group). As a result,

$$\mathbb{E}[\partial_i F(X) \partial_i F(X_t)] = \mathbb{E}\left[P_{t/2}(\partial_i F)(X)^2\right].$$

This proves the non-negativity property. One can also note that $\mathbb{E}\left[P_{t/2}(\partial_i F)(X)^2\right] - \mathbb{E}\left[P_{(t+s)/2}(\partial_i F)(X)^2\right]$ is the expectation of the variance of $P_{t/2}(\partial_i F)(X_{s/2})$ under the probability measure $\mathbb{P}[\bullet \mid X_0 = x]$ evaluated at $x = X$. This proves the monotonicity property. □

Let us note that exactly the same proof implies that for two functions $F, G$ we have

$$\mathrm{Cov}(F(X), G(X)) = \int_0^{+\infty} e^{-t} \sum_{1 \le i,j \le n} K(i,j) \mathbb{E}[\partial_i F(X) \partial_j G(X_t)] dt.$$

## 3.2 An application to the threshold function

Let us apply the formula to the threshold function of Gaussian field percolation. We first need to define a dynamics and discretize the field (since the formula is for Gaussian vectors). First, as above, we let $f_t = e^{-t} f + \sqrt{1 - e^{-2t}} \tilde{f}$ where $\tilde{f}$ is an independent copy of $f$. Consider the regular hexagonal lattice, whose dual is the regular triangular lattice $\mathbb{T}$. Given $\ell \in \mathbb{R}$ and $v \in \mathbb{T}$, color in black the hexagon centered at $v$ if $f(v) > \ell$ and in white if $f(v) < \ell$. Let $T_R$ be the supremum of all levels $\ell$ such that there is a black left-right crossing of $[-aR, aR] \times [-R, R]$. Also let $\mathbb{T}_R$ denote the set of sites of the triangular lattice that belong to $[-aR, aR] \times [-R, R]$. We study the Gaussian vector $(f(v))_{v \in \mathbb{T}_R}$ of covariance $K(v,w) = \kappa(v - w)$. Note that $\partial_v T_R = 1_{S_R = v}$ where $S_R$ is the coordinate $v$ such that $f(v) = T_R$. If $K$ has full rank (which we assume here) then such a $v$ is a.s. unique, therefore

$$\begin{aligned}\mathrm{Var}(T_R) &= \int_0^{+\infty} e^{-t} \sum_{v,w} K(v,w) \mathbb{P}[S_R(0) = v, S_R(t) = w]\, dt \\ &= \int_0^{+\infty} e^{-t} \mathbb{E}[\kappa(S_R(0) - S_R(t))]\, dt.\end{aligned}$$

W.s.w.h., the saddle point is also unique for the continuous model. So by applying the above formula on the hexagonal lattice multiplied by $\varepsilon$ and by letting $\varepsilon$ go to 0, we obtain that for the continuous percolation model on $f$ we also have

$$\mathrm{Var}(T_R) = \int_0^{+\infty} e^{-t} \mathbb{E}[\kappa(S_R(0) - S_R(t))]\, dt.$$

Note that this gives a uniform bound on $\mathrm{Var}(T_R)$: this quantity is less than or equal to $\kappa(0)$.

By the above, $\mathbb{E}[\kappa(S_R(0) - S_R(t))]$ is non-negative and non-increasing. Therefore,if $\kappa$ only takes positive values, then $\mathrm{Var}(T_R)$ goes to 0 if and only if, for every $t > 0$,

$\mathbb{E}\left[\kappa(S_R(0)-S_R(t))\right]$ goes to 0 i.e. if $S_R(0)-S_R(t)$ goes to $\infty$ in probability. Such a behavior in an example of a chaotic behavior as defined in a more general way in [Cha08].

Note also that this formula also implies that $g_\alpha$ cannot be in the universality class of percolation if $\alpha < 3/2$. Indeed, in this case we have $\mathbb{E}\left[\kappa(S_R(0)-S_R(t))\right] \geq cR^{-\alpha}$ and in the universality class of percolation we have $\mathrm{Var}(T_R) = R^{-3/2+o(1)}$ ([SW01]).

In this context, the inequality $\mathrm{Var}(T_R) \leq \kappa(0)$ is the Poincaré inequality:

$$\mathrm{Var}(F(X)) \leq \sum_{1\leq i,j\leq n} K(i,j)\mathbb{E}[\partial_i F(X)\partial_j F(X)].$$

In [Riv19] it is proved that $\mathrm{Var}(T_R)$ goes to 0 for $g_\alpha$, $\alpha > 2$ (with some strong hypotheses on $\kappa_\alpha$) by using a superconcentration formula (i.e. an improvement of the Poincaré inequality) in the i.i.d. case. This superconcentration formula (from [Tal94, CEL12]) is applied by considering a white noise decomposition of the field $f$ (which enables to have an i.i.d. process at our disposal). We do not give more details here.

### 3.3 An application to the number of connected components

Let us now apply the formula for the variance to the number of connected components. The problem is that this quantity does not satisfy the hypotheses of Proposition 3.1. By approximation, one can actually obtain the following result.

**Proposition 3.2.** *Let $X, K, X_t$ be as in Proposition 3.1 and assume furthermore that $K$ has full rank. Let $\ell \in \mathbb{R}$ and let $F, G : \mathbb{R}^n \to \mathbb{R}$ be two functions which are measurable with respect to $\sigma(\{x_i \geq \ell\}, i = 1, \cdots, n)$. If $x \in \mathbb{R}^n$, we let $x^{\oplus i} \in \mathbb{R}^n$ be the vector defined by $x_j^{\oplus i} = x_j$ if $j \neq i$ and $x_i^{\oplus i} = \ell + 1$ (or any other number larger than $\ell$). We define similarly $x^{\ominus i}$ but with $x_i^{\ominus i} = \ell - 1$. Then,*

$$\begin{aligned}\mathrm{Cov}(F(X), G(X)) = \int_0^{+\infty} e^{-t} &\sum_{1\leq i,j\leq n} K(i,j)\gamma_{(X(i),X_t(j))}(\ell,\ell) \\ &\mathbb{E}\left[\Big(F(X^{\oplus i}) - F(X^{\ominus i})\Big)\Big(G(X_t^{\oplus j}) - G(X_t^{\ominus j})\Big)\,\Big|\, X(i) = X_t(j) = \ell\right] dt\,,\end{aligned}$$

*where $\gamma_{(X(i),X_t(j))}$ is the density function of $(X(i), X_t(j))$ (note that $\gamma_{(X(i),X_t(j))}(\ell,\ell) \leq C/\sqrt{t}$ for some constant $C$ that depends only on $K$).*

*Proof.* The proof is by approximation. To understand how the above appears, consider some $C^1$ function $F$ that satisfies the hypotheses of the previous proposition and note that, if $X$ is non-degenerate, then

$$\mathbb{E}\left[\partial_i F(X)\right] = -\int_{\mathbb{R}^n} F(X)\partial_i\gamma_X(x)dx,$$

$\gamma_X$ is the density function of $X$. If we rather a consider a function $F$ which is measurable with respect to $\sigma(\{x_i \geq \ell\}, i = 1, \cdots, n)$, then

$$\begin{aligned} & -\int_{\mathbb{R}^n} F(X)\partial_i \gamma_X(x)dx \\ &= \int_{\mathbb{R}^{n-1}} \left(F(x^{\oplus i}) - F(x^{\ominus i})\right)\gamma_{X_i}(x_1, \cdots, x_{i-1}, \ell, x_{i+1}, \cdots, x_n)dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n \\ &\qquad = \mathbb{E}\left[F(X^{\oplus i}) - F(X^{\ominus i}) \mid X_i = \ell\right]\gamma_{X_i}(\ell). \end{aligned}$$

By approximation of $F, G$ by smooth functions and by a similar computation as above, we have the desired result. □

Note that, in the case of increasing events $A, B$ measurable with respect to $\sigma(\{x_i \geq \ell\}, i = 1, \cdots, n)$, Proposition 3.2 implies that

$$\begin{aligned} \mathrm{Cov}(1_{X\in A}, 1_{X\in B}) = \int_0^{+\infty} e^{-t} \sum_{1\leq i,j\leq n} & K(i,j)\gamma_{(X(i),X_t(j))}(\ell,\ell) \\ & \times \mathbb{P}\left[X \in \mathrm{Piv}_i(A),\, X_t \in \mathrm{Piv}_j(B) \mid X(i) = X_t(j) = \ell\right] dt\,, \end{aligned}$$

where $\mathrm{Piv}_i(A)$ is the pivotal event i.e. the event that changing the sign of $X_i - \ell$ changes the outcome of $A$. In particular, this implies the so-called FKG inequality: $\mathrm{Cov}(1_{X\in A}, 1_{X\in B}) \geq 0$ if the coefficients of $K$ are non-negative.

Consider the discrete Gaussian model on the triangular lattice considered in Subsection 3.2 and let $N_R(\ell)$ denote the number of connected components of the interface between black and white restricted to $[-R, R]^2$. Note that changing the color of one hexagon can change the value of $N_R$ by at most 2. So we obtain that

$$\mathrm{Var}(N_R) \leq \int_0^{+\infty} e^{-t} \sum_{v,w\in\mathbb{T}_R} |\kappa(v-w)| \frac{C}{\sqrt{t}} \times 2^2 dt = C' \sum_{v,w\in\mathbb{T}_R} |\kappa(v-w)|,$$

where $\mathbb{T}_R$ is the triangular grid restricted to $[-R, R]^2$. For the B-F field and for $g_\alpha$ with $\alpha > 2$, $\kappa$ is $L^1$, so the above is at most $CR^2$, which is the expected behavior. One direction in order to prove this upper-bound for the model in the continuum is to consider a grid of mesh $\varepsilon$ and estimate more precisely the quantities in the formula (without a careful study we just obtain that $\mathrm{Var}(N_R) \leq CR^2/\varepsilon^4$). Let us mention that an analogous study (but which is easier since the events that are considered live on disjoint domains) enables to prove the first part of the quasi-independence result Theorem 2.7 (and one can see that the exponent $\alpha > 4$ is indeed what one needs so that, if $B_R, B'_R$ are two boxes of size $R$ at distance $R$ from each other, $\sum_{v\in\mathbb{T}\cap B_R, w\in\mathbb{T}\cap B'_R} \kappa(v-w)$ goes to 0).

## 4 Some original motivations

In this section, we explain some motivations.

### 4.1 Hilbert's $16^{th}$ problem

In the case of the 2-dimensional sphere, Hilbert's $16^{th}$ problem is the following: Let $P \in \mathbb{R}_d^{hom}[x_0, x_1, x_2]$; describe $\{P = 0\} \cap \mathbb{S}_2$ (number of connected components? arrangement of the connected components?). The connected components (which are topological circles if there is no degeneracy) have a forest structure: can we describe this forest? The following has been proved by Harnack in 1876: the supremum of the number of connected components equals $d^2 - 3n + 4$. A general classification of the homogeneous polynomials according to Hilbert's problem seems out of reach and analogous questions have been studied for random models. For instance, it is proved in [GW11] that, if $P_d$ is a Kostlan polynomial of degree $d$, then the probability that the number of connected components of its zero set is of order $d^2 - 3n + 4$ is exponentially small. The local properties of the forest structure have been studied in in the case of Riemannian models (the result is essentially that locally any tree structure can be found) in [SW19, CS19]. Analogous questions have been studied in higher dimensions. In particular, it is proved that locally any topology can be found in the zero set of random models for algebraic and Riemannian models [GW14b, CS14].

### 4.2 Berry's and Bogomolny and Schmit's conjectures

Consider a compact Riemannian surface $S$ with negative curvature, let $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\triangle_M$ (it is known that $\lambda_j$ converges to $+\infty$) and let $(\varphi_j)_j$ denote a corresponding orthonormal $L^2$ basis (in particular, $\triangle_S \varphi_j = -\lambda_j \varphi_j$). Berrys conjecture [Ber77] states that the RPW is "a good modeling for the **determinictic** eigenfunctions $\varphi_j$". One way to state more precisely this is as follows (see [Ing17, ABM18] for more rigorous approaches): let $X$ be a random uniform point on $S$ and zoom at scale $1/\sqrt{\lambda_j}$ around $X$. This model converges to the RPW as $j \to +\infty$.

Motivated by this conjecture which suggests some universal behavior for the eigenfunctions of $\triangle_S$, Bogomolny and Schmit have conjectured that in some way RPW has the same behavior as Bernoulli percolation [BS02, BS07, BDS07]. This can be stated as follows (and the following conjecture is also believed to hold for the B-F field and the fields $g_\alpha$, $\alpha > 3/2$): consider a quad $Q$ (i.e. a Jordan domain with two distinguished disjoint segments on its boundary). Then, the probability that a nodal line crosses $RQ$ converges as $R \to +\infty$ to a conformal invariant quantity that is the same as for Bernoulli percolation (the fact that this quantity converges has only be proved in the case of Bernoulli percolation on the triangular lattice [Smi01]).

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