EXTENDING THE FKG INEQUALITY

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ABSTRACT. We quickly show how to extend the FKG inequality from finite volume to the setting of a continuous field. This in particular applies to continuous Gaussian fields with non-negative covariances.

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Date: October 22, 2019.

1. Setting

We consider a random element ϕ in $C(\mathbb{R}^d)$ (which carries the usual topology of uniform convergence on compact sets). We assume that for all increasing continuous $f, g\colon \mathbb{R}^k \to [0,\infty)$ and all $x_1, \dots, x_k \in \mathbb{R}^d$

$$\begin{aligned}\mathbb{E}(f(\phi(x_1), \dots, \phi(x_k))\, g(\phi(x_1), \dots, \phi(x_k))) \\ \geq \mathbb{E}f(\phi(x_1), \dots, \phi(x_k)) \cdot \mathbb{E}g(\phi(x_1), \dots, \phi(x_k)) \; .\end{aligned}$$

We call this the 'finite FKG' condition.

Example 1.1. By [Pit82], this holds for example, if $(\phi(x_1), \dots, \phi(x_k))$ are jointly Gaussian for every $x_1, \dots, x_k \in \mathbb{R}^d$ such that the for any $x, y \in \mathbb{R}^d$ we have $\mathrm{cov}(\phi(x), \phi(y)) \geq 0$.

2. Extension to continuous increasing functionals

Proposition 2.1. *If the finite FKG condition holds and $f, g\colon C(\mathbb{R}^d) \to [0,1]$ are continuous and increasing, then $\mathbb{E}(f(\phi)g(\phi)) \geq \mathbb{E}f(\phi) \cdot \mathbb{E}g(\phi)$.*

Proof. There are many ways to do this. Here is one: Let $\eta_\epsilon\colon \mathbb{R}^d \to [0,\infty)$ be a smooth compactly supported approximation of the identity and $u_R\colon \mathbb{R}^d \to [0,1]$ be smooth, have compact support and satisfy $u(x) = 1$ for all x with $|x| \leq R$. Fix $R, \epsilon > 0$ and set

$$\phi_n(x) = u_R(x) \cdot 2^{-nd} \sum_{v \in 2^{-n}\mathbb{Z}^d} \phi(v)\eta_\epsilon(x - v) \; .$$

By the finite FKG condition we obtain $\mathbb{E}(f(\phi_n)g(\phi_n)) \geq \mathbb{E}f(\phi_n)\mathbb{E}g(\phi_n)$ (observing that ϕ_n depends monotonically on only finitely many $\phi(v)$ values). It is not difficult to check that $\phi_n \to u_R \cdot \phi * \eta_\epsilon =: \psi_{R,\epsilon}$ as $n \to \infty$ uniformly on compact sets and since f and g are continuous, we get by dominated convergence

$$\mathbb{E}(f(\psi_{R,\epsilon})g(\psi_{R,\epsilon})) \geq \mathbb{E}f(\psi_{R,\epsilon})\mathbb{E}g(\psi_{R,\epsilon}) \; .$$

Also $\psi_{R,\epsilon} \to \phi * \eta_\epsilon$ as $R \to \infty$ and $\phi * \eta_\epsilon \to \phi$ as $\epsilon \to 0$ uniformly on compact sets and repeating the convergence argument above twice yields the required result. ☐

3. Extension to general increasing functionals

We can conclude using a similar argument as in [Pit82].

Theorem 3.1. *If the finite FKG condition holds and $f, g\colon C(\mathbb{R}^d) \to [0,\infty)$ are measurable and increasing, then $\mathbb{E}(f(\phi)g(\phi)) \geq \mathbb{E}f(\phi) \cdot \mathbb{E}g(\phi)$.*

Proof. Without loss of generality (by truncating, scaling and monotone convergence), f and g take values in $[0,1]$. Since $1/N \cdot \sum_{i=1}^{N} 1(f \geq i/N) \uparrow f$ as $N \to \infty$ and the $\{f \geq i/N\}$ are increasing sets, we may assume $f = 1_A$ with A increasing (and analogously for g). By Proposition 2.1, it suffices to prove that given f, we can find a sequence $f_n\colon C(\mathbb{R}^d) \to [0,1]$ of continuous increasing functions such that $f_n(\phi) \to f(\phi)$ a.s. Let $C_n \subset C(\mathbb{R}^d)$ and $K_n \subset A \subset U_n$ be such that C_n and K_n are compact, U_n is open, $\mathbb{P}(\phi \notin C_n) \leq 2^{-n}$ and $\mathbb{P}(\phi \in U_n \setminus K_n) \leq 2^{-n}$ (we are using regularity of the law of ϕ, which is automatic since it is a probability measures on a Polish space, twice here). The sets

$$B_r^R(\alpha_0) = \{\alpha \in C(\mathbb{R}^d)\colon \|\alpha - \alpha_0\|_{\bar{B}_R(0)} < r\}$$

form a basis of the topology where $\|\cdot\|_{\bar{B}_R(0)}$ is the supremum norm on $\bar{B}_R(0)$. For $\alpha_0 \in K_n \subset A$ the set $C_n \cap \{\alpha\colon \alpha \geq \alpha_0\}$ is compact and contained in the open set U_n so there are $r, R > 0$ such that

$$B_r^R(0) + C_n \cap \{\alpha\colon \alpha \geq \alpha_0\} \subset U_n \ .$$

Hence $C_n \cap \{\alpha\colon \alpha > \alpha_0 - r \text{ on } \bar{B}_R(0)\} \subset U_n$. K_n is compact, so we can pick $\alpha_1, \dots, \alpha_k \in K_n$ and $r_1, \dots, r_k, R_1, \dots, R_k > 0$ with $K_n \subset \cup_i B_{r_i}^{R_i}(\alpha_i)$ and

$$C_n \cap \{\alpha\colon \alpha > \alpha_i - 3r_i \text{ on } \bar{B}_{R_i}(0)\} \subset U_n \quad \forall i \ .$$

Let $r_0 = r_1 \wedge \cdots \wedge r_k$ and $R_0 = R_1 \vee \cdots \vee R_k$. By the above, for $\rho \in K_n$ one has $C_n \cap \{\alpha\colon \alpha \geq \rho - 2r_0 \text{ on } \bar{B}_{R_0}(0)\} \subset U_n$. Again, using compactness of K_n, take $\rho_1, \dots, \rho_l \in K_n$ such that $K_n \subset \cup_j \{\alpha\colon \alpha > \rho_j - r_0 \text{ on } \bar{B}_{R_0}(0)\}$. Finally, let $f_n\colon C(\mathbb{R}^d) \to [0,1]$ be

$$\begin{aligned} f_n(\rho) &= \max_j \ 0 \vee \Big(1 - r_0^{-1} \inf_{\psi \geq \rho_j - r_0} \|\psi - \rho\|_{\bar{B}_{R_0}(0)}\Big) \\ &= \max_j \ 0 \vee \Big(1 - r_0^{-1} \|((\rho_j - r_0) - \rho) \vee 0\|_{\bar{B}_{R_0}(0)}\Big) \ . \end{aligned}$$

It is easy to see that f_n is continuous, increasing and

$$\begin{aligned} 1(\rho \in K_n) &\leq 1(\exists j\colon \rho > \rho_j - r_0 \text{ on } \bar{B}_{R_0}(0)) \leq f_n(\rho) \\ &\leq 1(\exists j\colon \|((\rho_j - r_0) - \rho) \vee 0\|_{\bar{B}_{R_0}(0)} < r_0) \\ &= 1(\exists j\colon \rho > \rho_j - 2r_0 \text{ on } \bar{B}_{R_0}(0)) \leq 1(\rho \notin C_n) + 1(\rho \in U_n) \ . \end{aligned}$$

Since $K_n \subset A \subset U_n$ and $f = 1_A$, $\mathbb{E}|f(\phi) - f_n(\phi)| \leq \mathbb{P}(\phi \notin C_n) + \mathbb{P}(\phi \in U_n \setminus K_n) \leq 2^{1-n}$ and thus $f_n(\phi) \to f(\phi)$ a.s. as $n \to \infty$ as required. □

References

[Pit82] L. D. Pitt. 'Positively correlated normal variables are associated'. In: *Ann. Probab.* 10.2 (1982), pp. 496–499.