EXTENDING THE FKG INEQUALITY

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ABSTRACT. We quickly show how to extend the FKG inequality from finite volume to the setting of a continuous field. This in particular applies to continuous Gaussian fields with non-negative covariances.

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Date: October 22, 2019.
1. Setting

We consider a random element $\phi$ in $C(\mathbb{R}^d)$ (which carries the usual topology of uniform convergence on compact sets). We assume that for all increasing continuous $f, g: \mathbb{R}^k \to [0, \infty)$ and all $x_1, \ldots, x_k \in \mathbb{R}^d$

$$E(f(\phi(x_1), \ldots, \phi(x_k)) g(\phi(x_1), \ldots, \phi(x_k))) \geq E(f(\phi(x_1), \ldots, \phi(x_k)) \cdot E(g(\phi(x_1), \ldots, \phi(x_k))).$$

We call this the ‘finite FKG’ condition.

Example 1.1. By [Pit82], this holds for example, if $(\phi(x_1), \ldots, \phi(x_k))$ are jointly Gaussian for every $x_1, \ldots, x_k \in \mathbb{R}^d$ such that the for any $x, y \in \mathbb{R}^d$ we have $\text{cov}(\phi(x), \phi(y)) \geq 0$.

2. Extension to continuous increasing functionals

Proposition 2.1. If the finite FKG condition holds and $f, g: C(\mathbb{R}^d) \to [0, 1]$ are continuous and increasing, then $E(f(\phi) g(\phi)) \geq E(f(\phi)) \cdot E(g(\phi)).$

Proof. There are many ways to do this. Here is one: Let $\eta_\epsilon: \mathbb{R}^d \to [0, \infty)$ be a smooth compactly supported approximation of the identity and $u_\epsilon: \mathbb{R}^d \to [0, 1]$ be smooth, have compact support and satisfy $u(x) = 1$ for all $x$ with $|x| \leq R$. Fix $R, \epsilon > 0$ and set

$$\phi_n(x) = u_R(x) \cdot 2^{-nd} \sum_{v \in \mathbb{Z}^d} \phi(v) \eta_\epsilon(x - v).$$

By the finite FKG condition we obtain $E(f(\phi_n) g(\phi_n)) \geq E(f(\phi_n)) \cdot E(g(\phi_n))$ (observing that $\phi_n$ depends monotonically on only finitely many $\phi(v)$ values). It is not difficult to check that $\phi_n \to u_R \cdot \phi \ast \eta_\epsilon =: \psi_{R, \epsilon}$ as $n \to \infty$ uniformly on compact sets and since $f$ and $g$ are continuous, we get by dominated convergence

$$E(f(\psi_{R, \epsilon}) g(\psi_{R, \epsilon})) \geq E(f(\psi_{R, \epsilon})) \cdot E(g(\psi_{R, \epsilon})).$$

Also $\psi_{R, \epsilon} \to \phi \ast \eta_\epsilon$ as $R \to \infty$ and $\phi \ast \eta_\epsilon \to \phi$ as $\epsilon \to 0$ uniformly on compact sets and repeating the convergence argument above twice yields the required result. $\square$

3. Extension to general increasing functionals

We can conclude using a similar argument as in [Pit82].

Theorem 3.1. If the finite FKG condition holds and $f, g: C(\mathbb{R}^d) \to [0, \infty)$ are measurable and increasing, then $E(f(\phi) g(\phi)) \geq E(f(\phi)) \cdot E(g(\phi)).$
Proof. Without loss of generality (by truncating, scaling and monotone convergence), \( f \) and \( g \) take values in \([0, 1]\). Since \( 1/N \cdot \sum_{i=1}^{N} 1(f \geq i/N) \uparrow f \) as \( N \to \infty \) and the \( \{f \geq i/N\} \) are increasing sets, we may assume \( f = 1_A \) with \( A \) increasing (and analogously for \( g \)). By Proposition 2.1, it suffices to prove that given \( f \), we can find a sequence \( f_n: C(\mathbb{R}^d) \to [0, 1] \) of continuous increasing functions such that \( f_n(\phi) \to f(\phi) \) a.s. Let \( C_n \subset C(\mathbb{R}^d) \) and \( K_n \subset A \subset U_n \) be such that \( C_n \) and \( K_n \) are compact, \( U_n \) is open, \( \mathbb{P}(\phi \notin C_n) \leq 2^{-n} \) and \( \mathbb{P}(\phi \in U_n \setminus K_n) \leq 2^{-n} \) (we are using regularity of the law of \( \phi \), which is automatic since it is a probability measures on a Polish space, twice here). The sets

\[
B_R^R(\alpha_0) = \{ \alpha \in C(\mathbb{R}^d) : \|\alpha - \alpha_0\|_{B_R(0)} < r \}
\]

form a basis of the topology where \( \| \cdot \|_{B_R(0)} \) is the supremum norm on \( B_R(0) \). For \( \alpha_0 \in K_n \subset A \) the set \( C_n \cap \{ \alpha : \alpha \geq \alpha_0 \} \) is compact and contained in the open set \( U_n \) so there are \( r, R > 0 \) such that

\[ B_R^R(\alpha_0) + C_n \cap \{ \alpha : \alpha \geq \alpha_0 \} \subset U_n. \]

Hence \( C_n \cap \{ \alpha : \alpha > \alpha_0 - r \) on \( B_R(0) \} \subset U_n. \) \( K_n \) is compact, so we can pick \( \alpha_1, \ldots, \alpha_k \in K_n \) and \( r_1, \ldots, r_k, R_1, \ldots, R_k > 0 \) with \( K_n \subset \bigcup_i B_{R_i}^R(\alpha_i) \) and

\[
C_n \cap \{ \alpha : \alpha > \alpha_i - 3r_i \) on \( B_{R_i}(0) \} \subset U_n \end{equation}\]

Let \( r_0 = r_1 \wedge \cdots \wedge r_k \) and \( R_0 = R_1 \lor \cdots \lor R_k \). By the above, for \( \rho \in K_n \) one has \( C_n \cap \{ \alpha : \alpha \geq \rho - 2r_0 \) on \( B_{R_0}(0) \} \subset U_n. \) Again, using compactness of \( K_n \), take \( \rho_1, \ldots, \rho_l \in K_n \) such that \( K_n \subset \bigcup_j \{ \alpha : \alpha > \rho_j - r_0 \) on \( B_{R_0}(0) \}. \) Finally, let \( f_n: C(\mathbb{R}^d) \to [0, 1] \) be

\[
f_n(\rho) = \max_j 0 \lor \left( 1 - r_0^{-1} \inf_{\psi \geq \rho_j - r_0} \|\psi - \rho\|_{B_{R_0}(0)} \right) = \max_j 0 \lor \left( 1 - r_0^{-1} \|((\rho_j - r_0) - \rho) \lor 0\|_{B_{R_0}(0)} \right) .
\]

It is easy to see that \( f_n \) is continuous, increasing and

\[
1(\rho \in K_n) \leq 1(\exists j: \rho > \rho_j - r_0 \) on \( B_{R_0}(0) \} \leq f_n(\rho) \leq 1(\exists j: \|((\rho_j - r_0) - \rho) \lor 0\|_{B_{R_0}(0)} < r_0) \]

\[
= 1(\exists j: \rho > \rho_j - 2r_0 \) on \( B_{R_0}(0) \} \leq 1(\rho \notin C_n) + 1(\rho \in U_n) .
\]

Since \( K_n \subset A \subset U_n \) and \( f = 1_A, \mathbb{E}[f(\phi) - f_n(\phi)] \leq \mathbb{P}(\phi \notin C_n) + \mathbb{P}(\phi \in U_n \setminus K_n) \leq 2^{1-n} \) and thus \( f_n(\phi) \to f(\phi) \) a.s. as \( n \to \infty \) as required. \( \square \)

REFERENCES