Uncertainty quantification for nonlinear hyperbolic PDEs

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Lituya Bay, Alaska

Lituya Bay Mega Tsunami

- Took place on July 9, 1958.
- Magnitude 7.8 Earthquake along Fairweather fault (Alaska).
- Triggered massive Rock slide of 3×10^7 m^3 volume.
- Wave run-up to shore of 525 m !!!
- Maximum Wave height of $50 - 80$ m !!!
- Most powerful tsunami ever recorded.

Lituya Bay Post-Tsunami

What is needed I: The model

- Two-layer Savage-Hutter (Shallow water) model.

$$
\begin{cases}
\frac{\partial h_1}{\partial t} + \frac{\partial q_1}{\partial x} = 0 \\
\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_1^2}{h_1} + \frac{g}{2} h_1^2\right) + g h_1 \frac{\partial h_2}{\partial x} = g h_1 \frac{dH}{dx} + S_f + S_{b_1} \\
\frac{\partial h_2}{\partial t} + \frac{\partial q_2}{\partial x} = 0 \\
\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_2^2}{h_2} + \frac{g}{2} h_2^2\right) + r g h_2 \frac{\partial h_1}{\partial x} = g h_2 \frac{dH}{dx} - r S_f + S_{b_2} + \tau.
\end{cases}
\tag{1}
$$

- With
  - Coulomb friction: $\tau = -g(1-r) h_2 \frac{q_2}{|q_2|} \tan(\delta_0)$,
  - Interlayer friction: $S_f = c_f \frac{h_1 h_2}{h_2 + r h_1}(u_2 - u_1)|u_2 - u_1|$

What is needed II: The numerical scheme

- Savage-Hutter equations are Non-conservative hyperbolic system

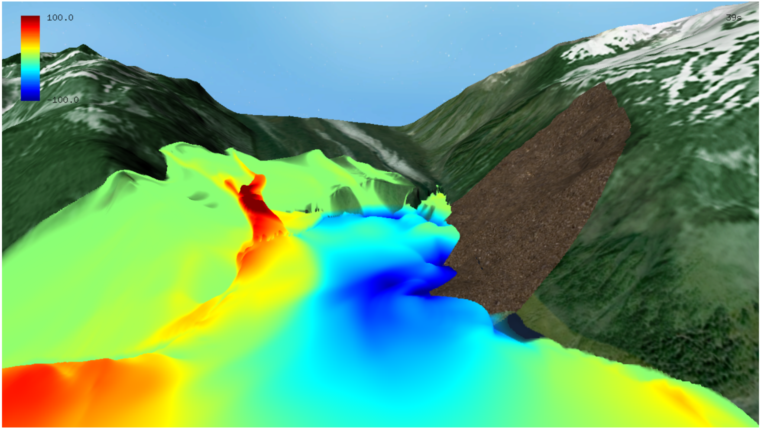
$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_x = 0.$$

- Specially designed Path conservative finite volume scheme
- Need to discretize Non-conservative product carefully.
- Optimized GPU implementation.

What is needed III: Inputs

- Initial data.
- Boundary conditions.
- Model parameters:
  - Acceleration due to gravity g.
  - Interlayer density ratio r
  - Bottom friction parameters $S_{b_{1,2}}$
  - Coulomb friction angle δ_0
  - Interlayer friction parameter c_f

Run-up at $T = 39s$



Critique of the simulation

- Sources of Errors
  - Modeling error
    Savage-Hutter is a good model (checked in the lab).
  - Numerical (discretization) error.
    - Good numerical scheme (Discretization error can be made as small as possible).
  - Measurement (Data) errors:
    - Rather low for initial data and boundary conditions.
    - Unacceptably high for r, c_f, δ_0 (even in the lab !!!)
    - Standard deviation is more than 50 percent of mean !!!
- High measurement error $\Rightarrow$ low trust in simulation ?

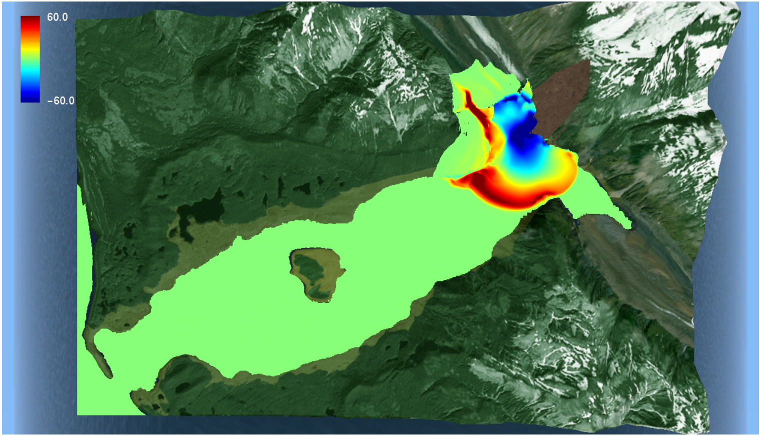
Generic situation in Science and Engineering

- Mathematical modeling of any physical/chemical/biological phenomena:
- Model inputs: are obtained by Measurements:
  - Initial conditions.
  - Boundary data.
  - Coefficients.
  - Parameters.
- Measurements are Uncertain.
- Uncertain Inputs $\Rightarrow$ Uncertain Solutions (Outputs).
- $+$ Many models based on Uncertain Dynamics (high Model $+$ Numerical error).

What is UQ

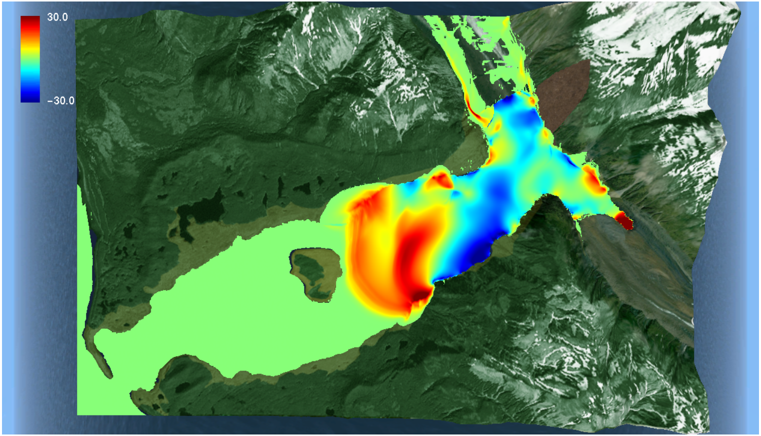
- Uncertainty quantification includes:
  - Modeling of uncertain inputs and dynamics.
  - Efficient Computation of the resulting output uncertainty.
  - Interpretation of the uncertain output.

Run-up Mean at $T = 39s$

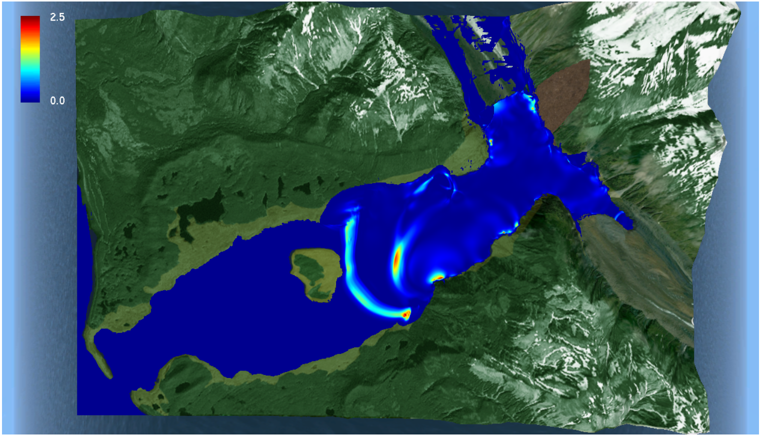


Run-up Variance at $T = 39s$

Run-up Mean at $T = 120s$



Run-up Variance at $T = 120s$



UQ for nonlinear hyperbolic PDEs

$$
\begin{aligned}
\mathbf{U}_t + \mathrm{div}(\mathbf{F}(k(x,t),\mathbf{U})) &= S(x,t,\mathbf{U}),\\
\mathbf{U}(x,0) &= \mathbf{U}_0(x),\\
\mathbf{U}|_{\partial D} &= \mathbf{U}_b(x,t).
\end{aligned}
$$

- Uncertainty in determining:
  - Flux Coefficients (Equations of state, Material properties of porous media)
  - Initial data (Initial wave displacement in tsunamis)
  - Source terms (Bottom topography in shallow water waves)
  - Boundary data (Plasma circuit breakers)
- UQ: Given uncertainty in inputs $\Rightarrow$ Compute uncertainty in the solution.

Issues

- How to model uncertainty in inputs ??
- Mathematical framework for uncertain solutions.
- Efficient numerical methods for UQ.

Modeling Input Uncertainty

- Use the Probabilistic framework a la Kolmogorov.
- Complete Probability space:
  - Ω (Set of Outcomes)
  - Σ (σ-algebra (field) of Events)
  - $\mathbb{P} : \Omega \mapsto [0, 1]$ with $\mathbb{P}(\Omega) = 1$ (Probabilty measure).

Random fields

- Use Random fields to model Uncertain:
  - Initial data.
  - Boundary conditions.
  - Fluxes.
  - Sources.
- $(\Omega, \Sigma, \mathbb{P})$ is a complete probability space.
- Random field $\mathbf{U} : (\Omega, \Sigma) \mapsto (\mathcal{F}, \mathcal{B}(\mathcal{F}))$ measurable
- $\mathcal{F}$ is a function space (separable Banach space) with Borel σ-algebra $\mathcal{B}(\mathcal{F})$
- For $\omega \in \Omega$, $\mathbf{U}(\omega) \in \mathcal{F}$.
- Example: Random initial data (scalar conservation laws):

$$u_0 : (\Omega, \Sigma) \mapsto (L^1(\mathbb{R}^d), \mathcal{B}(L^1(\mathbb{R}^d)))$$
$$u_0(., \omega) \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d), \mathbb{P} - a.s.$$

Representation of Random fields I: Parametric representation

- Random field represented by a finite number of parameters (Random Variables).
- Example I: Euler equations – Sod Shock tube – Uncertain initial location + amplitude:

$$
\mathbf{U}_0(x,\omega) = \begin{cases} \mathbf{U}_l + \alpha(\omega) & \text{if} \quad x \le \beta(\omega), \\ \mathbf{U}_r & \text{if} \quad x > \beta(\omega), \end{cases}
$$

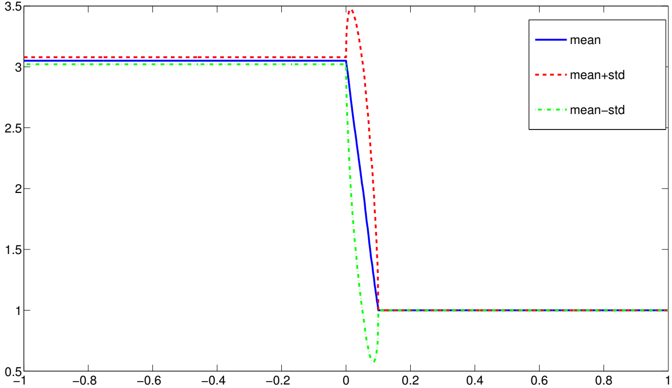
$$
\alpha \sim 0.05\mathcal{U}[-1,1]
$$

$$
\beta \sim 0.2\mathcal{U}[-1,1]
$$

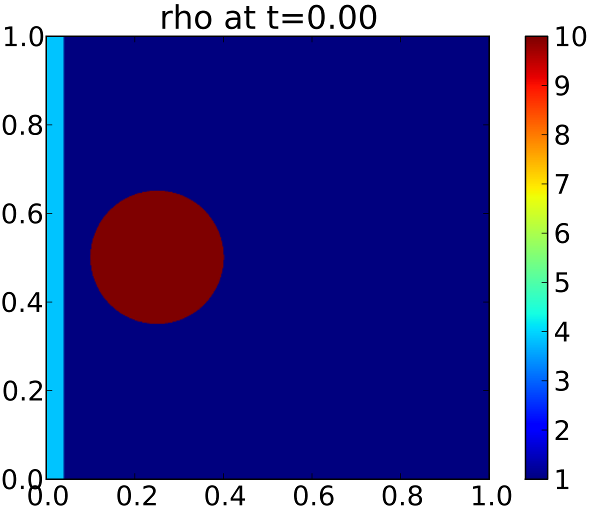
- 2 Uniformly distributed random parameters.

Euler equations – Sod Shock tube – Uncertain initial location + amplitude

- Mean $\pm$ Standard deviation.

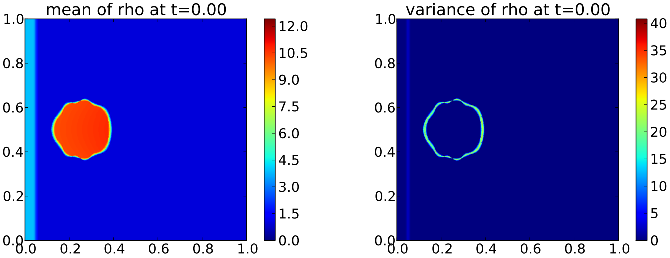


- Deterministic Initial data:



Ex II: Euler equations - Cloud shock interaction

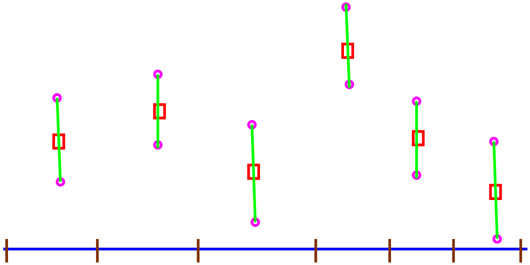
- Uncertain initial data in terms of 11 uniformly distributed parameters:



- Uncertainty in Shock location, amplitude, Bubble location, amplitude and geometry.

Ex III: Shallow water equations– bottom topography

- Real data bottom topography given by Digital Terrain Models.
- Typical representation:

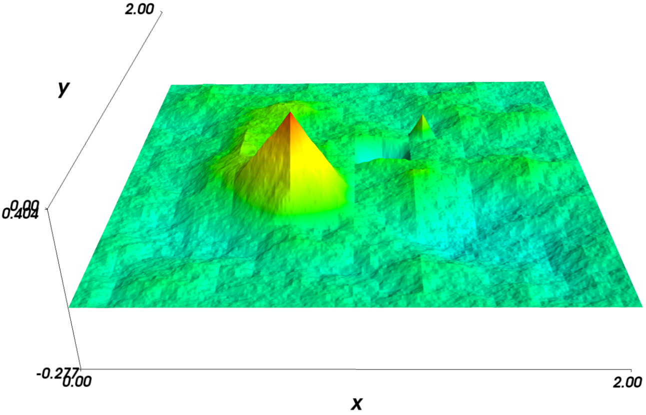


- Interpolation using hierarchical hat basis (SM, Schwab, Sukys, 2013)

Bottom topography: one sample (realization)

Hierarchical hat basis representation

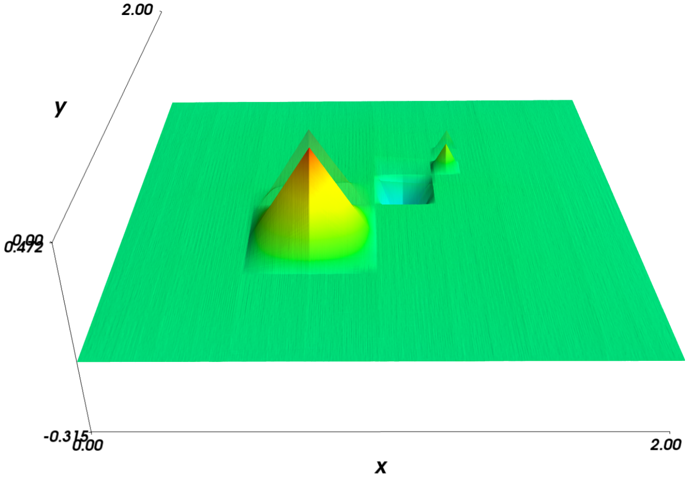
- 962 Random parameters !!!



Bottom topography: mean and standard deviation

Hierarchical hat basis representation

- 962 Random parameters !!!



Random fields II: Karhunen-Loeve expansions

- Bi-orthogonal decomposition (a la Fourier Series).
- General form of KL expansion:

$$f = \overline{f} + \sum \sqrt{\lambda_k} Z_k f_k.$$

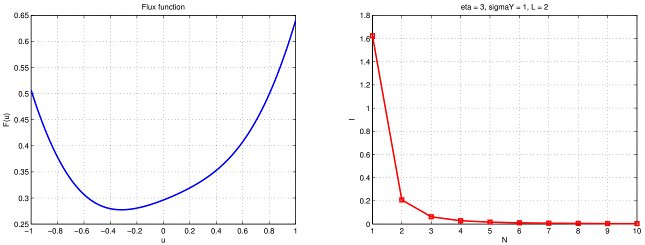
- Z_k's are Uncorrelated random variables as $\mathbb{E}(Z_i Z_j) := \lambda_j \delta_{ij}$.
- λ_k, f_k are eigenvalues (vectors) of the Covariance operator: $K_C : L^2(D) \mapsto L^2(D)$:

$$K_{C_f}[g](x) = \int_D C_f(x,y) g(y) dy, \, C_f(x,y) := \mathbb{E}(f(x,\omega) f(y,\omega)).$$

Ex I: Perturbed Burgers' flux

- Has the KL expansion:

$$f(\omega; u) = f(\mathbf{y}; \mathbf{u})\Big|_{\mathbf{y}=\mathbf{Y}(\omega)} = \frac{\mathbf{u}^2}{2} + \delta\Big(\sum_{\mathbf{j}\geq 1} \mathbf{y_j}\sqrt{\lambda_\mathbf{j}}\mathbf{\Phi_j}(\mathbf{u})\Big),$$



- Represented as a Gaussian process with exponential covariance: $C_Y(u_1, u_2) = \sigma_Y^2 e^{-|u_1 - u_2|/\eta}$

Ex II: Rock permeability for seismic imaging

- Seismic Acoustic pulses modeled by Wave equation:

$$p_{tt} + div(\mathbf{c}\nabla p) = 0.$$

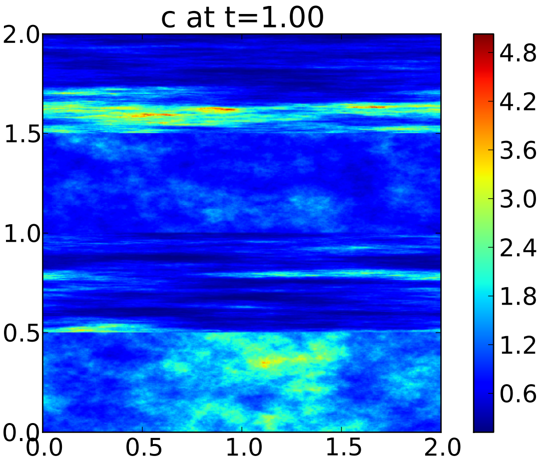
- Rewritten as a linear system of conservation laws.
- $\mathbf{c}$ is the rock permeability coefficient
- Highly uncertain – modeled by a log normal Gaussian random field:

$$\log(\mathbf{c}(x,\omega)) := \log(\overline{c}(x)) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} Z_k(\omega) g_k(x).$$

- Many different Covariance functions.
- Need Spectral FFT + Upscaling for efficient generation.

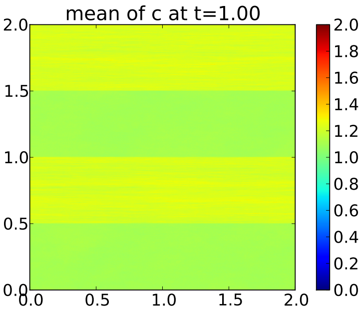
Ex II: 2-D log normal layered permeability field (sample)

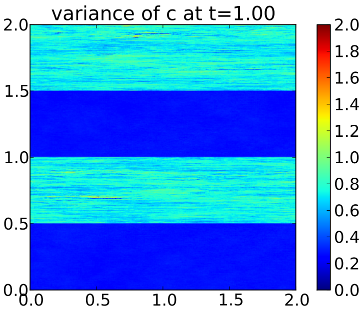
- $\approx$ 1000 uncertain parameters !!!



Ex II: 2-D log normal layered permeability field (statistics)

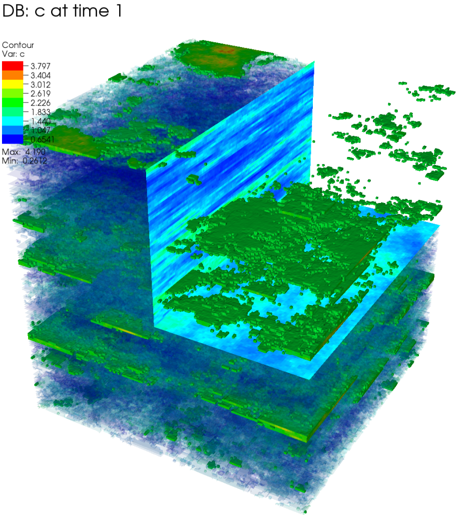
- ≈ 1000 uncertain parameters !!!





Ex II: 3-D log normal layered permeability field (sample)

- $\approx 10^6$ uncertain parameters !!!



Mathematical framework (scalar case)

- Random scalar conversation laws:

$$u_t(x,t,\omega) + \mathrm{div}(f(\omega; u(x,t,\omega))) = 0.$$
$$u(x,0,\omega) = u_0(x,\omega).$$

- with initial data and flux:

$$u_0 : (\Omega,\Sigma) \mapsto (L^1(\mathbb{R}^d), \mathcal{B}(L^1(\mathbb{R}^d)))$$
$$f : (\Omega,\Sigma) \mapsto (C^1(\mathbb{R}^1;\mathbb{R}^d); \mathcal{B}(C^1(\mathbb{R};\mathbb{R}^d)))$$

Random entropy solution

- Solution is a random field that satisfies,
  - Measurability: $u : \Omega \in \omega \mapsto u(x,t;\omega)$ is measurable from (Ω, Σ) to $C((0,T); L^1(\mathbb{R}^d))$.
  - Weak solution: u satisfies the integral identity:

$$\int_{\mathbb{R}^d \times \mathbb{R}_+} \left(u(x,t,\omega)\varphi_t(x,t) + \langle f(\omega; u(x,t,\omega), \nabla\varphi(x,t)\rangle \right) dxdt + \int_{\mathbb{R}^d} u(x,0,\omega)\varphi(x,0)dx = 0.$$

    for $\mathbb{P}$-a.e $\omega \in \Omega$.
  - Entropy conditions: satisfied for all entropy-entropy flux pairs and for $\mathbb{P}$-a.e $\omega \in \Omega$.

Well-posedness theorem: SM, Schwab, 2010, SM et al 2012.

- For sufficiently regular u_0,:
  - Existence: There exists a unique random entropy solution

$$u : \Omega \ni \omega \mapsto C_b(0, T; L^1(\mathbb{R}^d))$$

  - Construction:

$$u(\cdot, t; \omega) = S(t) u_0(\cdot, \omega) , \quad t > 0, \ \omega \in \Omega$$

  - Stability: $\mathbb{P}$-a.s $\omega \in \Omega$,

$$\begin{aligned}
\|u\|_{L^k(\Omega; C(0,T; L^1(\mathbb{R}^d)))} &\leq \|u_0\|_{L^k(\Omega; L^1(\mathbb{R}^d))} , \\
\|S(t)\, u_0(\cdot, \omega)\|_{(L^1 \cap L^\infty)(\mathbb{R}^d)} &\leq \|u_0(\cdot, \omega)\|_{(L^1 \cap L^\infty)(\mathbb{R}^d)} \\
TV(S(t) u_0(\cdot, \omega)) &\leq TV(u_0(\cdot, \omega))
\end{aligned}$$

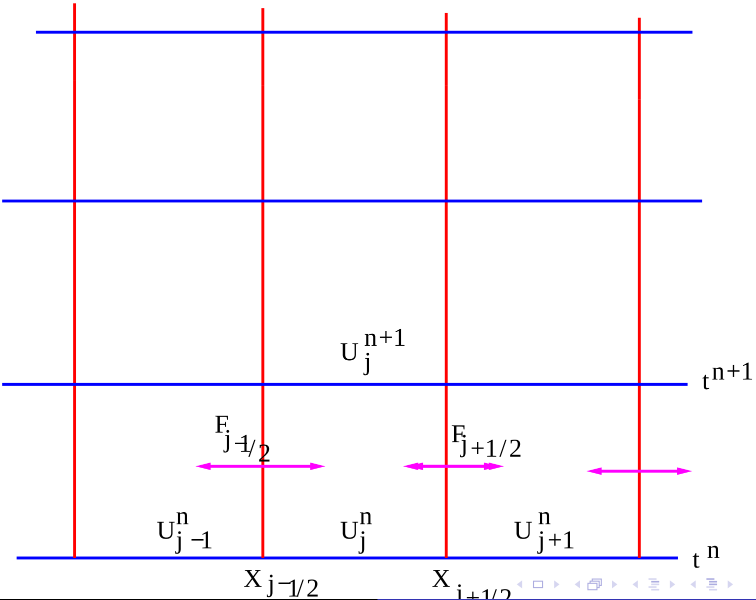
Schemes for random conservation laws

- Conservation law with uncertain initial data:

$$
\begin{aligned}
u_t(x,t,\omega) + \mathrm{div}(f(u(x,t,\omega))) &= 0. \\
u(x,0,\omega) &= u_0(x,\omega).
\end{aligned}
$$

- Discretization of Physical space-time.
- Standard Finite volume method

Finite volume Grid



Standard FVM

- Of the form:

$$u_j^{n+1} - u_j^n + \frac{\Delta t}{\Delta x}(F_{j+1/2} - F_{j-1/2}) = 0$$

- Have the following convergence rate:

$$\|u(.,t) - u_\tau(.,t)\|_{L^1(\mathbb{R}^d)} \leq C\Delta x^s.$$

- Work estimate:

$$\mathrm{Work}_\tau = \mathcal{O}(\Delta x^{-(d+1)}).$$

- Accuracy vs. Work:

$$\|u(.,t) - u_\tau(.,t)\|_{L^1(\mathbb{R}^d)} \leq C(\mathrm{Work}_\tau)^{-\frac{s}{d+1}}.$$

Schemes for random conservation laws

- Random conservation law:

$$u_t(x,t,\omega) + \mathrm{div}(f(\omega; u(x,t,\omega))) = 0.$$
$$u(x,0,\omega) = u_0(x,\omega).$$

- Need to discretize the probability space.
- Statistical sampling methods: Monte Carlo (**MC**) method.

MCFVM

- The MC algorithm:
  - Draw M i.i.d samples for the initial data and flux: $\{u_0^i, f^i\}_{1 \leq i \leq M}$.
  - For each sample: Solve conservation law by FVM to obtain u_τ^i.
  - Sample statistics:

$$\mathcal{M}^1 u(\cdot, t) \approx E_M[u_\tau(\cdot, t)] := \frac{1}{M} \sum_{i=1}^{M} u_\tau^i(\cdot, t).$$

$$\mathcal{M}^k u(t_1, \ldots, t_k) := \frac{1}{M} \sum_{i=1}^{M} \underbrace{(u_\tau^i(\cdot, t_1) \otimes \cdots \otimes u_\tau^i(\cdot, t_k))}_{k-\text{times}}.$$

MCFVM: Properties, SM, Schwab, 2010.

- Convergence:

$$\|\mathbb{E}[u(\cdot,t)] - E_M[u_\tau(\cdot,t;\omega)]\|_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq C_{\mathrm{stat}} M^{-\frac{1}{2}} + C_{\mathrm{st}} \Delta x^s.$$

- Number of samples: $M = \mathcal{O}(\Delta x)^{-2s}$.
- Accuracy vs. Work:

$$\|\mathbb{E}[u(\cdot,t)] - E_M[u_\tau(\cdot,t;\omega)]\|_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq C(\mathrm{Work}_\tau)^{-\frac{s}{d+1+2s}}.$$

- Slow convergence $\Rightarrow$ very high computational cost.

Multi-level Monte Carlo (MLMC) FVM:

- Heinrich 1995: Quadrature.
- Giles 2002: Stochastic ODEs.
- Barth, Schwab, Zollinger 2010: Elliptic PDEs.

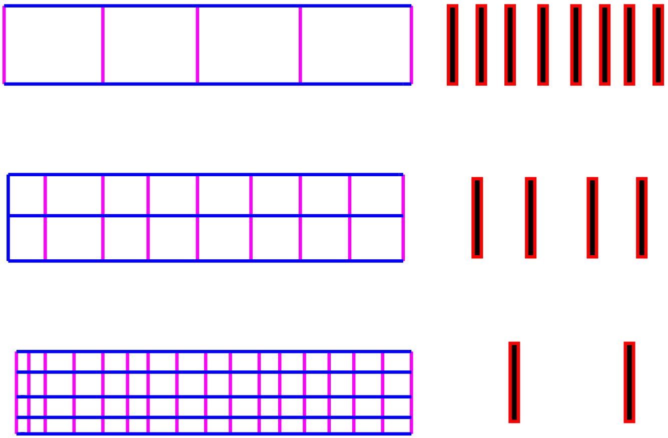
Multi-level Monte Carlo (MLMC) FVM: SM, Schwab, 2010

- MLMCFVM algorithm:
  - Different nested levels of resolution: l.
  - Draw M_l i.i.d samples for the initial data: $\{u^i_{l,0}\}_{1 \leq i \leq M_l}$.
  - For each draw: Solve conservation law by FVM to obtain $u^i_{l,\tau}$.
  - Sample statistics: with $u_{\tau,-1} = 0$,

$$\mathcal{M}^1 u(\cdot,t) \approx E^L[u(\cdot,t)] = \sum_{\ell=0}^{L} E_{M_\ell}[u_{\tau,\ell}(\cdot,t) - u_{\tau,\ell-1}(\cdot,t)]$$

$$\mathcal{M}^k u(t_1,\dots,t_k) := \sum_{\ell=0}^{L} E_{M_\ell}[u^{(k)}_{\tau,\ell}(\cdot,t) - u^{(k)}_{\tau,\ell-1}(\cdot,t)]$$

MLMCFVM



MESH Resolution

Number of samples

MLMCFVM: Properties

- Convergence:

$$\|\mathbb{E}[u(\cdot,t)] - E^L[u_\tau(\cdot,t,\omega)]\|_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq C_1 \Delta x_L^s + C_3 M_0^{-\frac{1}{2}} + C_2 \left\{ \sum_{\ell=0}^{L} M_\ell^{-\frac{1}{2}} \Delta x_\ell^{\frac{s}{2}} \right\}$$

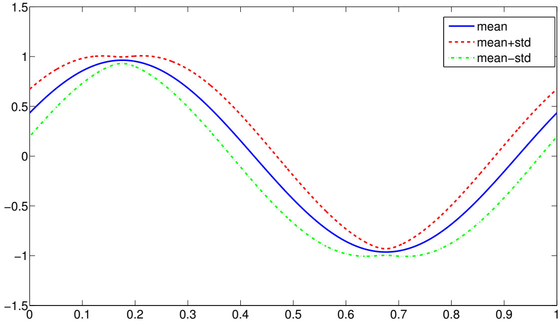
- Level dependent number of samples: $M_l = \mathcal{O}\left(\frac{\Delta x_l^s}{\Delta x_L^{2s}}\right)$
- Accuracy vs. Work: If $0 \leq s < (d+1)$,

$$\|\mathbb{E}[u(\cdot,t)] - E^L[u_\tau(\cdot,t;\omega)]\|_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq C(\mathrm{Work})^{-\frac{s}{d+1+s}}$$

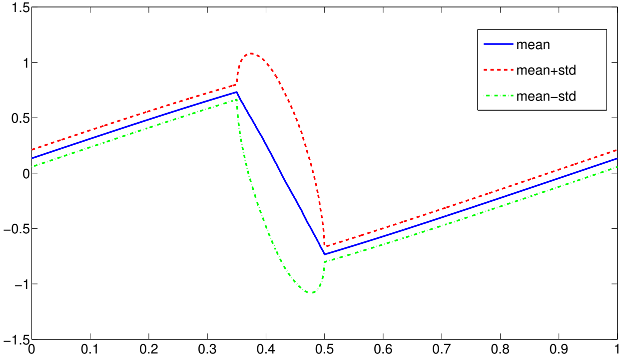
- Significantly more efficient than MCFVM !!!
- Sparse tensor higher moments computation with same efficiency.

1-D Burgers' with uncertain initial phase

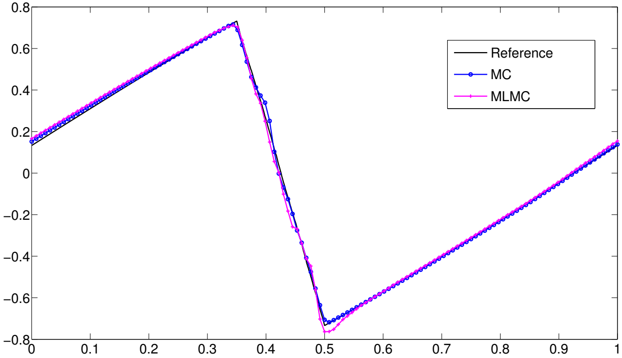
- 1 random parameter.



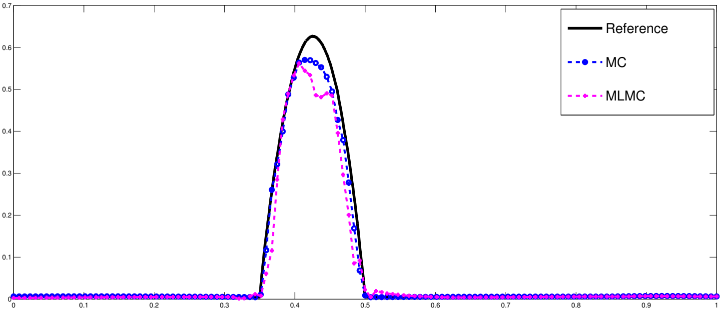
Mean ± Standard deviation



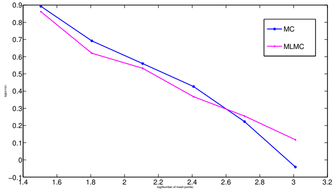
Mean: MC vs MLMC



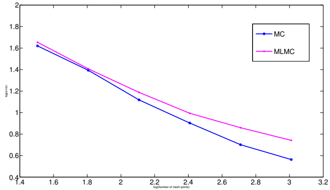
Variance: MC vs MLMC



log(resolution) vs. log(relative error)

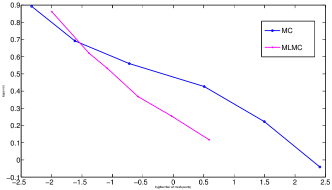


(a) mean

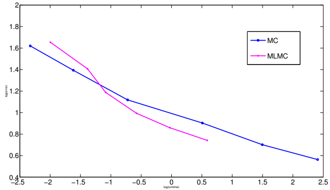


(b) variance

log(runtime) vs. log(relative error)



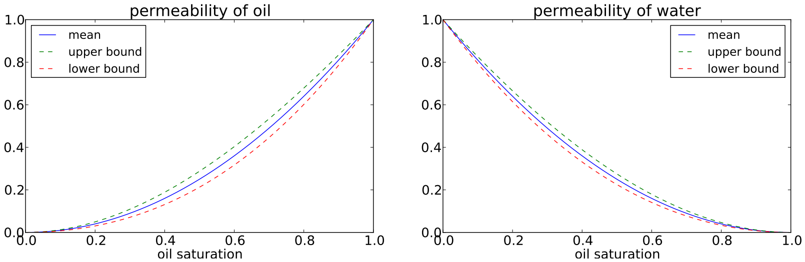
(c) mean



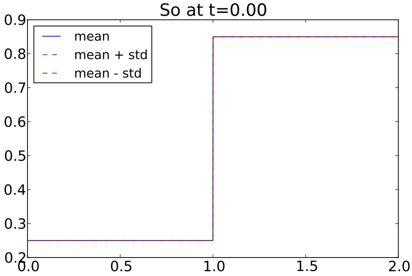
(d) variance

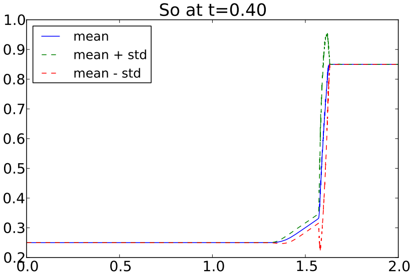
Buckley Leverette with uncertain relative permeabilities

- 2 random parameters.

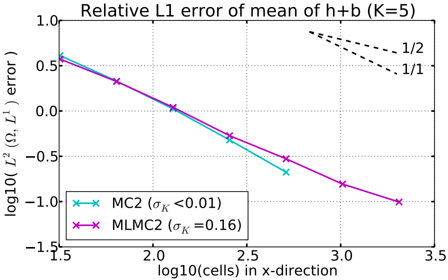


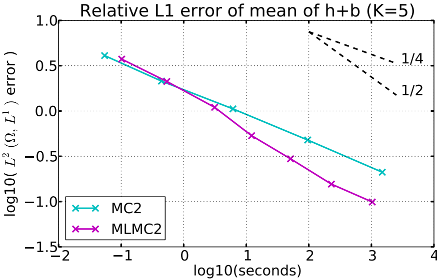
Buckley Leverette: mean $\pm$ std of Water saturation



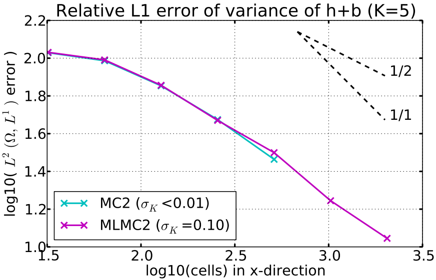


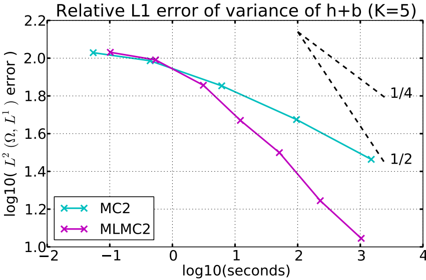
Buckley Leverette: convergence of mean





Buckley Leverette: convergence of variance





Linear systems of conservation laws

- Random linear symmetrizable systems of conservation laws:

$$\mathbf{U}_t(x,t,\omega) + \sum_{r=1}^{d} \frac{\partial}{\partial \mathbf{x}_r}\Big(\mathbf{A}_r(\mathbf{x},\omega)\mathbf{U}\Big) = 0.$$

$$\mathbf{U}(x,0,\omega) = \mathbf{U}_0(x,\omega).$$

- with uncertain initial data an flux:

$$\mathbf{U}_0 : (\Omega,\Sigma) \mapsto (L^2(\mathbf{D}), \mathcal{B}(L^2(\mathbf{D}))$$
$$\mathbf{A}_r : (\Omega,\Sigma) \mapsto (C^1(\mathbf{D})^{m\times m}; \mathcal{B}(C^1(\mathbf{D})^{m\times m})$$

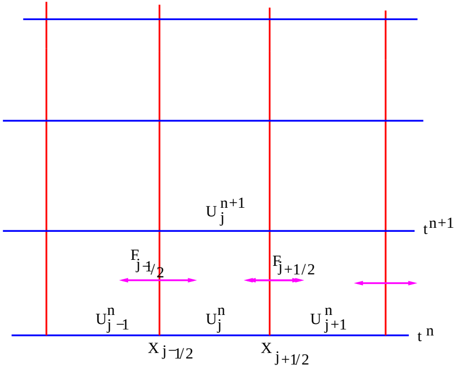
Random Weak solution

- Solution is a random field that satisfies,
  - Measurability: $\mathbf{U} : \Omega \in \omega \mapsto \mathbf{U}(x, t; \omega)$ is measurable from (Ω, Σ) to $C((0, T); L^2(\mathbf{D}))$.
  - Weak solution: $\mathbf{U}$ satisfies the integral identity:

$$\int\limits_{\mathbb{R}^d \times \mathbb{R}_+} \left(\mathbf{U} \cdot \varphi_t + \sum_{r=1}^{d} \mathbf{A}_r \mathbf{U} \cdot \frac{\partial}{\partial \mathbf{x}_r} \varphi \right) d\mathbf{x}dt + \int\limits_{\mathbb{R}^d} \mathbf{U}_0 \cdot \varphi(t=0)\ d\mathbf{x} = 0.$$

  for $\mathbb{P}$-a.e $\omega \in \Omega$.
- THM (SM, Schwab, Sukys 2014): Random weak solutions exist and are unique.

- Standard Finite volume method to discretize Space-time.



- Under suitable assumptions on initial data + coefficients $\mathbf{A}_r$, FVM Convergence rate:

$$\|\mathbf{U} - \mathbf{U}^{\Delta x}\|_{L^2} \leq C \Delta x^s$$

Schemes for Linear systems II: MCFVM

- The MC algorithm:
  - Draw M i.i.d samples for the initial data and flux: $\{\mathbf{U}_0^i, \mathbf{A}_r^i\}_{1 \leq i \leq M}$.
  - For each sample: Solve linear system by FVM to obtain $\mathbf{U}_\tau^i$.
  - Sample statistics:

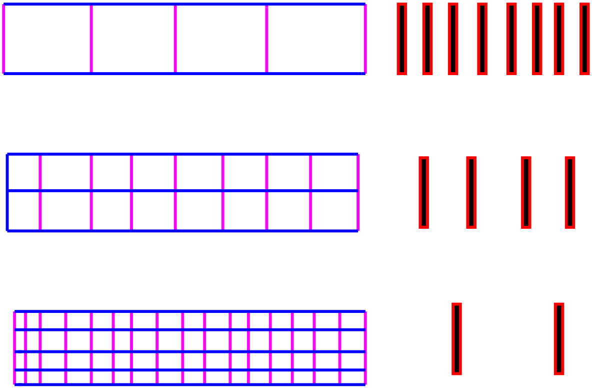
$$\mathbb{E}(\mathbf{U}(\cdot, t)) \approx E_M[\mathbf{U}_\tau(\cdot, t)] := \frac{1}{M} \sum_{i=1}^{M} \mathbf{U}_\tau^i(\cdot, t).$$

- Convergence (SM,Schwab,Sukys,2014):

$$\|\mathbb{E}[\mathbf{U}(\cdot, t)] - E_M[\mathbf{U}_\tau(\cdot, t; \omega)]\|_{L^2(\Omega; L^2(\mathbf{D}))} \leq C_{\mathrm{stat}} M^{-\frac{1}{2}} + C_{\mathrm{st}} \Delta x^s.$$

- Slow convergence $\Rightarrow$ very high computational cost.

Schemes for Linear systems III: MLMCFVM-SM,Schwab,Sukys 2014



MESH Resolution

Number of samples

MLMCFVM for linear systems

- Convergence:

$$\|\mathbb{E}[\mathbf{U}(\cdot,t)] - E^L[\mathbf{U}_\tau(\cdot,t,\omega)]\|_{L^2(\Omega;L^2(\mathbb{R}^d))} \leq C_1 \Delta x_L^s + C_3 M_0^{-\frac{1}{2}} + C_2 \left\{ \sum_{\ell=0}^{L} M_\ell^{-\frac{1}{2}} \Delta x_\ell^s \right\}$$

- Level dependent number of samples: $M_l = \mathcal{O}\left(\frac{\Delta x_l^{2s}}{\Delta x_L^{2s}}\right)$
- Same complexity as deterministic FVM !!!

Ex : Seismic imaging

- Seismic Acoustic pulses modeled by Wave equation:

$$p_{tt} + div(\mathbf{c}\nabla p) = 0.$$

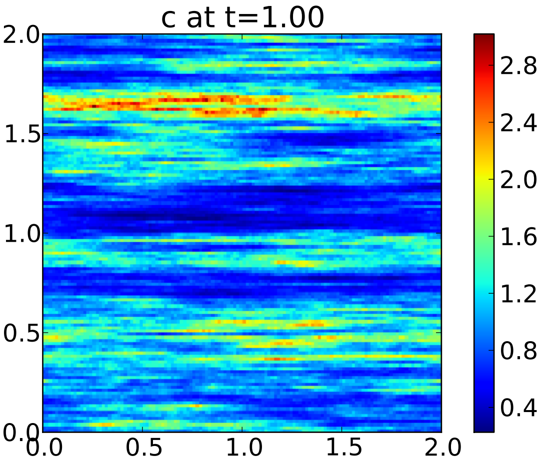
- Rewritten as a linear system of conservation laws.
- $\mathbf{c}$ is the rock permeability coefficient
- Highly uncertain – modeled by a log normal Gaussian random field:

$$\log(\mathbf{c}(x,\omega)) := \log(\overline{c}(x)) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} Z_k(\omega) g_k(x).$$

- Many different Covariance functions.
- Need Spectral FFT + Upscaling for efficient generation.

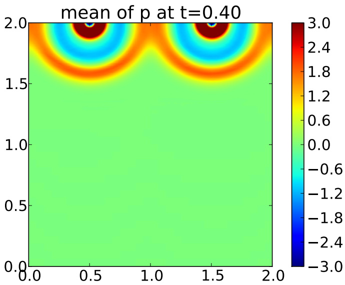
Ex : 2-D log normal layered permeability field (sample)

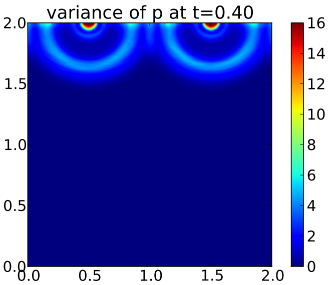
- ≈ 1000 uncertain parameters !!!



Ex : 2-D log normal layered permeability field $T = 0.4$

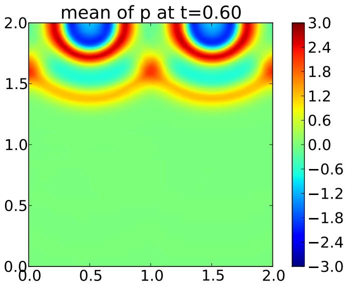
- $\approx$ 1000 uncertain parameters !!!

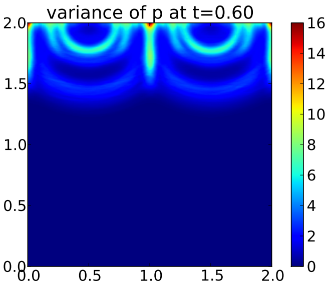




Ex : 2-D log normal layered permeability field $T = 0.6$

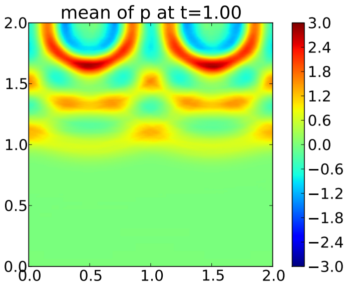
- $\approx$ 1000 uncertain parameters !!!





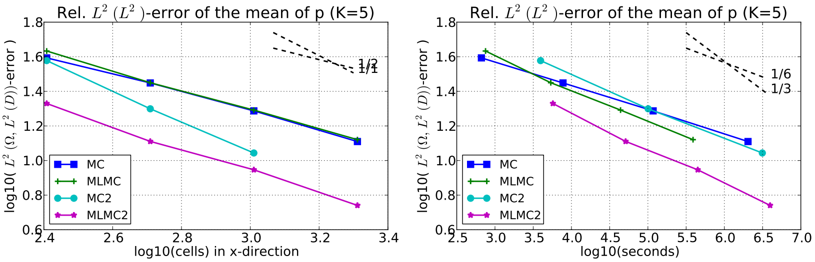
Ex : 2-D log normal layered permeability field $T = 1.0$

- ≈ 1000 uncertain parameters !!!



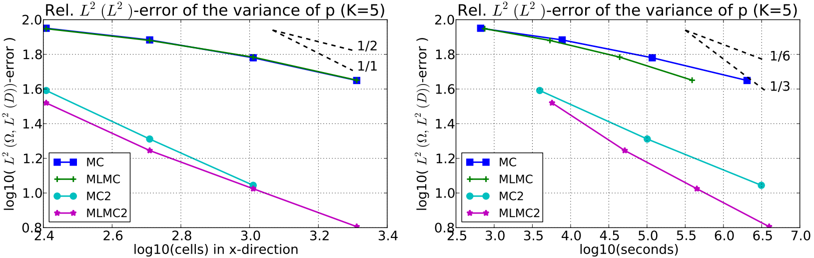
Convergence of mean

- ≈ 1000 uncertain parameters !!!



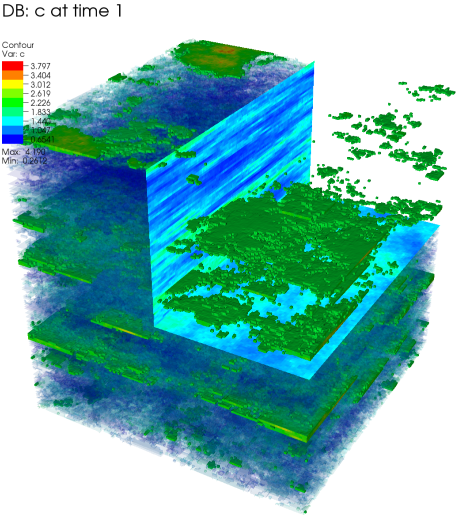
Convergence of variance

- $\approx$ 1000 uncertain parameters !!!



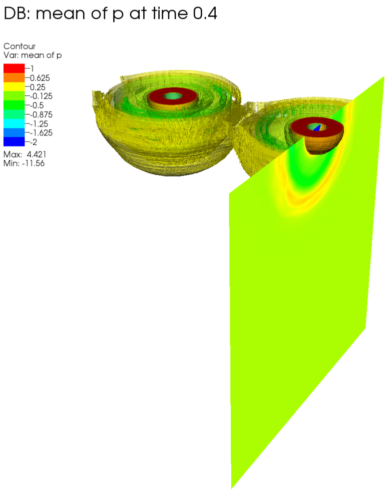
Ex II: 3-D log normal layered permeability field (sample)

- $\approx 10^6$ uncertain parameters !!!



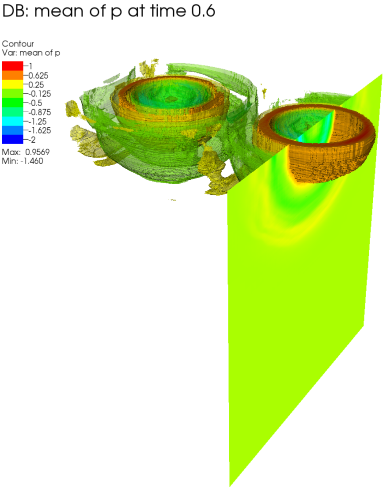
Ex II: Mean at $T = 0.4$

- $\approx 10^6$ uncertain parameters !!!



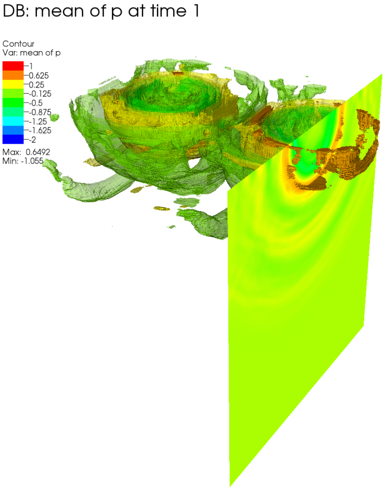
Ex II: Mean at $T = 0.6$

- $\approx 10^6$ uncertain parameters !!!



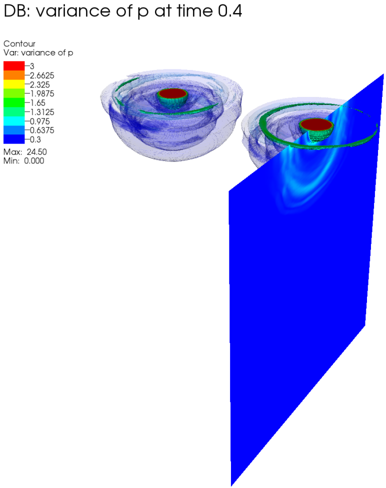
Ex II: Mean at $T = 1.0$

- $\approx 10^6$ uncertain parameters !!!



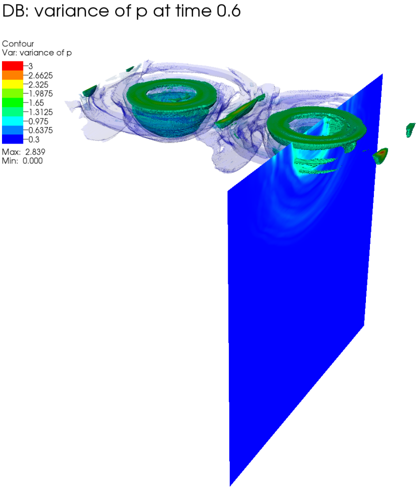
Ex II: Variance at $T = 0.4$

- $\approx 10^6$ uncertain parameters !!!



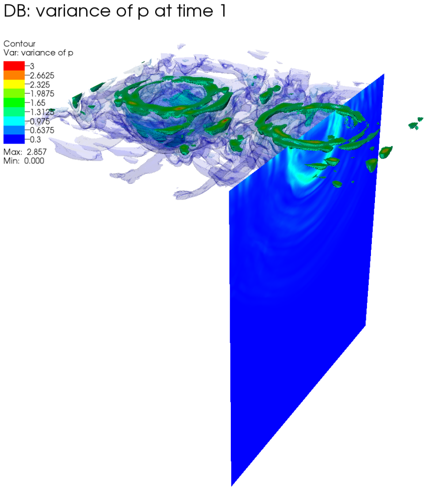
Ex II: Variance at $T = 0.6$

- $\approx 10^6$ uncertain parameters !!!

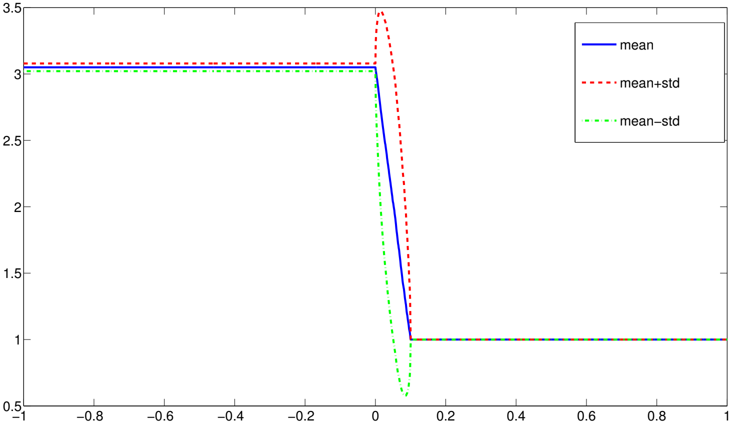


Ex II: Variance at $T = 1.0$

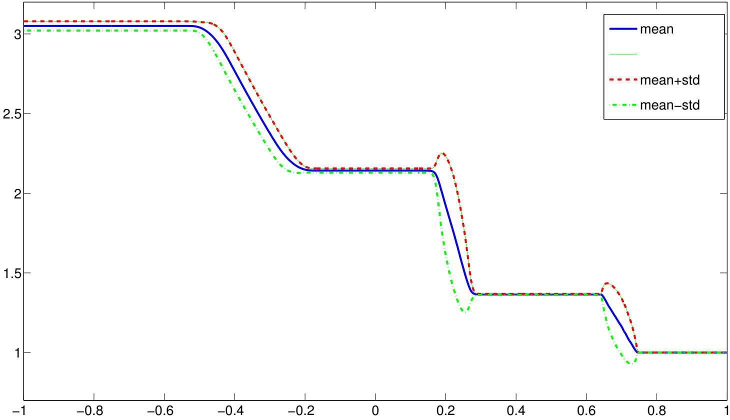
- $\approx 10^6$ uncertain parameters !!!



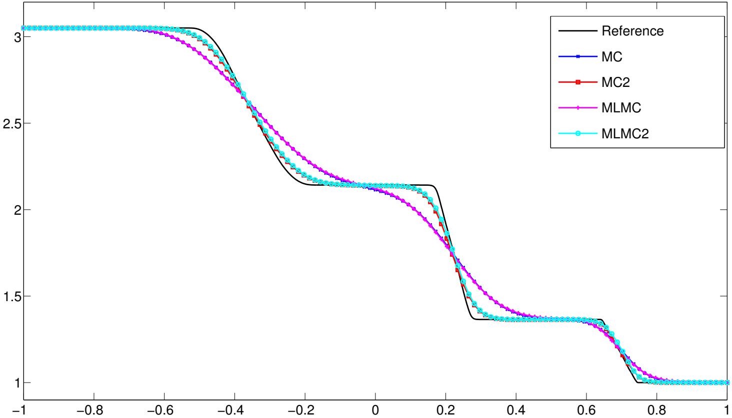
Euler equations with uncertain shock location and amplitude



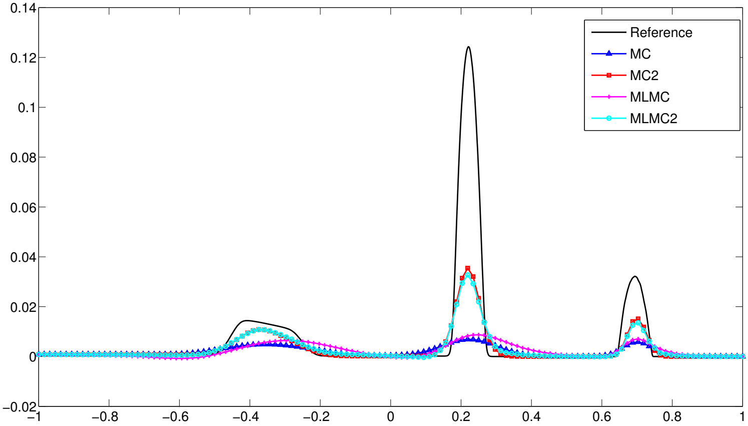
Mean $\pm$ Standard deviation



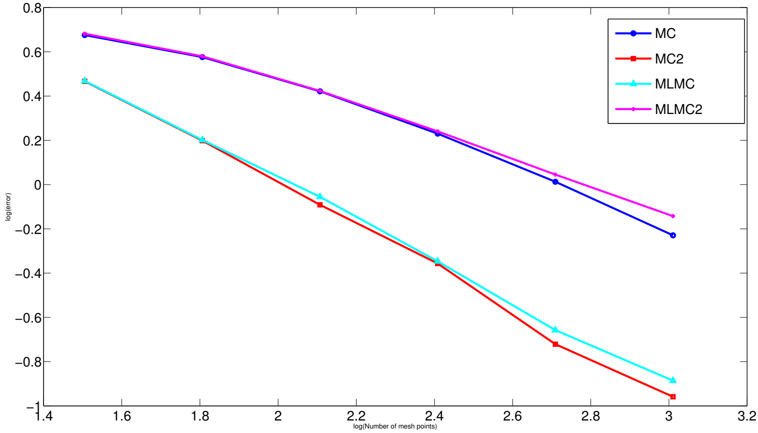
Mean: MC vs MLMC



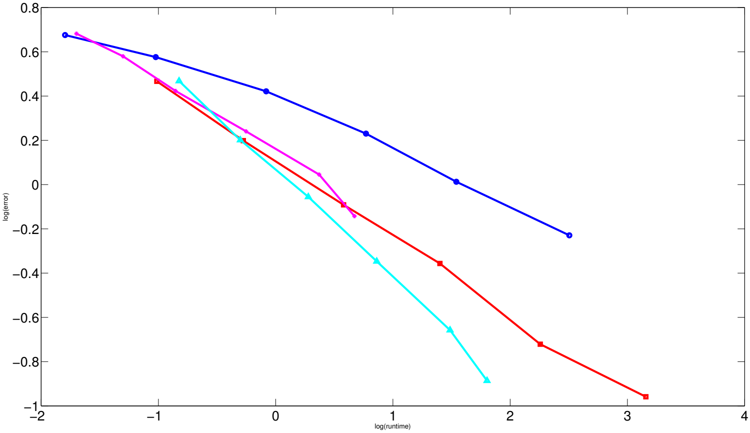
Variance: MC vs MLMC



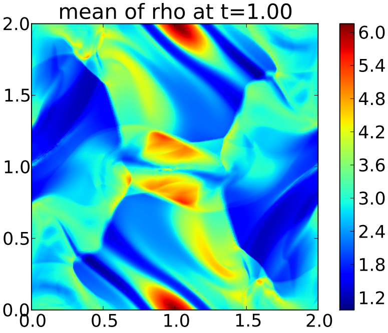
log(resolution) vs. log(relative error in mean)

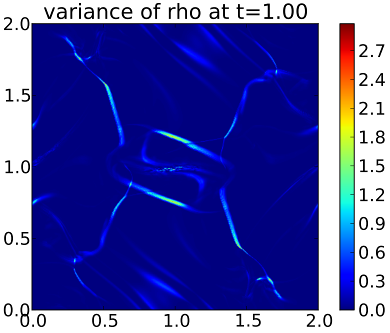


log(runtime) vs. log(relative error in mean)

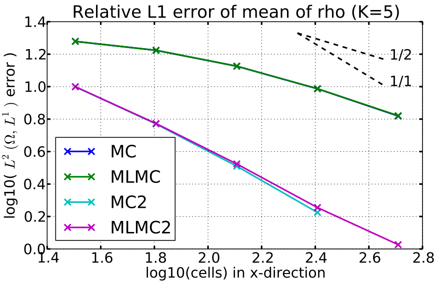


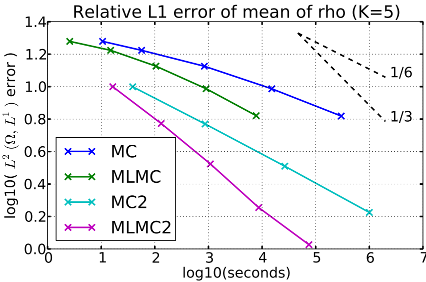
Uncertain Orszag-Tang vortex for MHD (2 Sources of uncertainty)



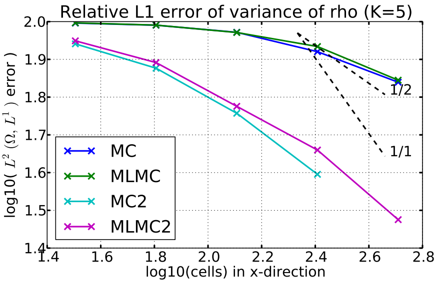


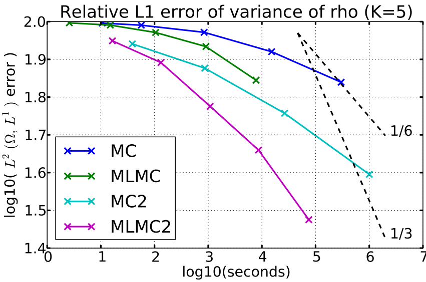
Uncertain Orszag-Tang vortex for MHD (Convergence of mean)



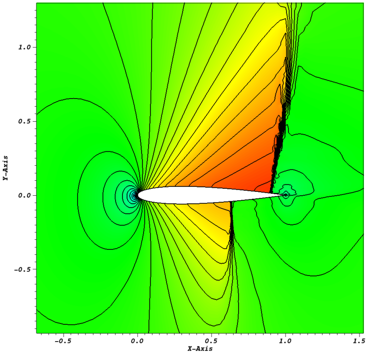


Uncertain Orszag-Tang vortex for MHD (Convergence of variance)

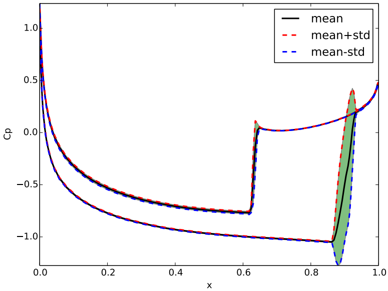


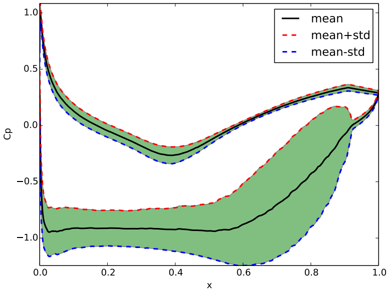


Flow past aerofoils: Left: NACA0012, Right:RAE2826



Mean$\pm$ STD for C_p: Left: NACA0012, Right:RAE2826





Variance decay: Left: NACA0012, Right:RAE2826