Mathematical Finance

Exercise sheet 1

Exercise 1.1 Let $(M_n)_{n\in\mathbb{N}}$ be a martingale such that $M_0 = 0$ and

$$|M_n - M_{n-1}| \leq a_n \quad P\text{-a.s.}$$

for each n and a sequence (a_n) of non-negative constants, with $\sum_{i=1}^{\infty} a_i^2 = A^2 < \infty$.

(a) Prove that M is bounded in L^2. Deduce that $M_n \to M_\infty$ almost surely and in L^2, for some M_∞ in L^2.

(b) Show that

$$P\left(\sup_{k\geq 0} M_k \geq c\right) \leq \exp\left(-\frac{c^2}{2A^2}\right),$$

for any $c > 0$.

Hint: Try applying Doob's maximal inequality to $(e^{\lambda M_n})$, for some $\lambda > 0$. You may use the inequality $\cosh(x) \leq e^{x^2/2}$ (for $x \in \mathbb{R}$).

Exercise 1.2 Let $\mathbb{S}$ denote the family of simple predictable processes H, i.e.

$$H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}$$

for stopping times $0 = \tau_0 < \tau_1 < ... < \tau_{i+1} < \infty$ and bounded $\mathcal{F}_{\tau_i}$-measurable H_i for $i = 0, 1, ..., n+1$. Let $\mathbb{D}$ denote the family of adapted càdlàg processes and $\mathbb{L}$ denote the family of adapted càglàd processes on $[0, \infty)$. We endow $\mathbb{D}$ and $\mathbb{L}$ with the topology of convergence uniformly on compacts in probability, generated by the metric

$$d(X, Y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \mathbb{E}[|(X - Y)|_k^* \wedge 1].$$

Moreover, let the space of all measurable random variables L^0 be endowed with the topology generated by convergence in probability. Show that:

(a) The vector spaces $\mathbb{L}$ and $\mathbb{D}$ are complete.

(b) For some càdlàg process X, the following are equivalent:

1. The map $J_X : \mathbb{S} \to \mathbb{D}$ with $J_X(H) := H_0 X_0 + \sum_{i=1}^{n} H_i (X_{\tau_{i+1}\wedge\cdot} - X_{\tau_i \wedge \cdot})$, for $H \in \mathbb{S}$, is continuous with respect to the u.c.p. metric on $\mathbb{S}$ and $\mathbb{D}$, in other words, X is a good integrator.
2. For every $t \in [0, \infty)$, the mapping $I_{X^t} : \mathbb{S} \to L^0$ with $I_{X^t}(H) := J_X(H)_t$, for $H \in \mathbb{S}$, is continuous with respect to the uniform norm metric on $\mathbb{S}$.

Exercise 1.3 Prove that the set of good integrators is a vector space and show that it is generically not closed with respect to the ucp topology. Construct in particular examples of processes which are not good integrators but can be approximated by good integrators.

Exercise 1.4 Let μ be a probability measure on $(0,+\infty)$. Consider (on some probability space) independent $N, Y_1, Y_2, Y_3, ...$ where each Y_i has distribution μ and $N = (N_t)_{t\in[0,1]}$ is a Poisson process on $[0,1]$ of rate $\lambda > 0$. Consider the compound Poisson process X on $[0,1]$ given by

$$X_t = \sum_{i=1}^{N_t} Y_i.$$

(a) Find a necessary and sufficient condition for X to be a submartingale with respect to its natural filtration.

(b) Show that under that condition, X is a submartingale of class (D). Find a decomposition

$$X_t = M_t + A_t \quad \forall t \in [0,1],$$

where M is a martingale and A is an increasing predictable process, both with càdlàg trajectories.

(c) Show through direct calculations that X is a good integrator.