Mathematical Finance

Exercise sheet 10

Exercise 10.1 Consider a financial market S. Let

$$\mathcal{X}_1 = \{1 + \vartheta \cdot S : \vartheta \in \Theta^1_{\text{adm}}\}.$$

Show that $\mathcal{X}_1$ satisfies the switching property: for any $X \in \mathcal{X}_1$ and any strictly positive $X' \in \mathcal{X}_1$, a stopping time τ and event $A \in \mathcal{F}_\tau$, we have that

$$\tilde{X} = \mathbb{1}_{\Omega \setminus A} X_\cdot + \mathbb{1}_A \frac{X'_{\tau \vee \cdot}}{X'_\tau} X_{\tau \wedge \cdot}$$

belongs to X_1.

For the following questions, we consider a financial market satisfying (NFLVR), with a numéraire (which we set to be equal to 1) and a d-dimensional risky asset with discounted prices S_t taking values in $D \subseteq \mathbb{R}^d$.

We work under an ESM, Q, and we will work with polynomial models. Consider the following two models:

- The Black-Scholes model:

$$dS_t = S_t \sigma dW_t \quad (S_0 \in \mathbb{R}^d_{>0}),$$

  where $\sigma \in \mathbb{R}^{d \times d}$ is invertible, and W is a d-dimensional Brownian motion. S_t in the right-hand side is viewed as a diagonal matrix with entries S^i.

- The SABR model (in Bachelier form):

$$dS^1_t = Y_t dW_t, \quad dS^2_t = \alpha S^2_t dB_t \quad (S^1_0, S^2_0 > 0),$$

  for some parameter $\alpha > 0$ and Brownian motions W, B with fixed correlation $\rho \in [-1, 1]$. S^2 is the stochastic volatility, which we assume that we can trade through forward volatility contracts.

For these two models, solve the following exercises:

Exercise 10.2 Show that S is a polynomial process, in other words, that for any $s \leq t$ and any polynomial p of degree n we have

$$E[p(S_t) \mid \mathcal{F}_s] = q(S_s),$$

where q is a polynomial of degree $\leq n$, whose coefficients are functions of $t - s$.

Exercise 10.3 Find the delta hedge for a payoff of the form $p(S_t)$, for p a polynomial.

Exercise 10.4 Note that the models under consideration are Markovian. Define and compute the transition semigroup $(P_t)_{t \geq 0}$ (under Q), as it acts on the set of real polynomials $\mathrm{Pol}(\mathbb{R}^d)$ ($\mathbb{R}^2$ in the case of the SABR model).

Show that in this setting,

$$P_{t-s}f(X_s) = (\nabla P_{t-\cdot}f(S_\cdot) \bullet S)_s + P_t(x), \quad Q^x\text{-a.s.},$$

where Q^x is the law of S started from a given point x.

Show that this equality also holds P-almost surely, where P is the historical measure.

Exercise 10.5 (Python)

Implement the discretised delta hedge from exercise 3, for the payoff $H = (S_T)^3$. Compute the error between the hedge and the payoff, as well as the difference between the payoff and a hedging strategy that only trades in S, but not Y.