Mathematical Finance

Exercise sheet 11

Exercise 11.1 Let U be a utility function satisfying the Inada conditions, i.e. $U \in C^1(\mathbb{R}_+;\mathbb{R})$ is strictly increasing, strictly concave and

$$U'(0) := \lim_{x \searrow 0} U'(x) = +\infty$$
$$U'(+\infty) := \lim_{x \to +\infty} U'(x) = 0.$$

Let J be the Legendre transform of $-U(-\cdot)$,

$$J(y) := \sup_{x>0}(U(x) - xy),$$

and denote by $I := (U')^{-1}$ the inverse of the derivative of U.

Show the following properties:

1. J is strictly decreasing and strictly convex.
2. $J'(0) = -\infty$, $J'(+\infty) = 0$, $J(0) = U(+\infty)$ and $J(+\infty) = U(0)$.
3. For any $x > 0$,
$$U(x) = \inf_{y>0}(J(y) + xy).$$
4. For any $y > 0$,
$$J(y) = U(I(y)) - yI(y).$$
5. $J' = -I$.

Exercise 11.2 Let the financial market $S = (S_k)_{k=0,\dots,N}$ be defined over the *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,\dots,N}, P)$ and satisfy $\mathcal{M}^a(S) \neq \emptyset$, and let U be a utility function satisfying the Inada conditions. Consider the value functions

$$u(x) = \sup_{X_T \in C(x)} E[U(X_T)] \text{ and } v(y) = \inf_{Q \in \mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where V is the convex conjugate of U and

$$C(x) = \{X_T \in L^0(\Omega, \mathcal{F}_T, P) \mid \forall Q \in \mathcal{M}^a(S) : E_Q[X_T] \leq x\}.$$

Show that the optimisers $\hat{X}_T(x)$, $\hat{Q}(x)$ and $\hat{y}(x)$ satisfy $U'\left(\hat{X}_T(x)\right) = \hat{y}(x)\frac{d\hat{Q}(x)}{dP}$ for each $x \in \mathrm{dom}(U)$.

Exercise 11.3 Let $C \subseteq L^0_+$ be convex, closed and bounded in probability, and assume that $1 \in C$. Let $D := \{z \in L^0_+ \mid \forall f \in C : \ E[zf] \leq 1\}$. Show that for any $g \in L^\infty$,

$$\inf\{x \in \mathbb{R}_+ \mid \exists f \in C : \ xf \geq g\} = \sup_{z \in D} E[zg].$$