Mathematical Finance

Exercise sheet 4

Let $\mathcal{S}$ denote the set of semimartingales and $\mathbb{S}_1 := \{H \in \mathbb{S} : ||H||_\infty \leq 1\}$ the unit ball of simple predictable processes. The Emery topology is a topology on $\mathcal{S}$ generated by the metric

$$d_E(X,Y) := \sum_{n=1}^{\infty} 2^{-n} \sup_{H \in \mathbb{S}_1} E\left[1 \wedge \sup_{t \leq n} |(H \bullet (X-Y))_t|\right].$$

Exercise 4.1 Show that

(a) $\mathcal{S}$ endowed with the Emery topology is a topological vector space.

(b) $\mathcal{S}$ is closed in the Emery topology and complete with respect to the metric d_E.

Exercise 4.2 Show that the Emery topology is invariant under an equivalent change of measure.

Exercise 4.3 Let the set of adapted càglàd processes $\mathbb{L}$ be endowed with the u.c.p. topology and the set of semimartingales $\mathcal{S}$ be endowed with the Emery topology, and let X be a semimartingale. Show that

$$J_X : \mathbb{L} \ni Y \mapsto (Y \bullet X) \in \mathcal{S}$$

is continuous.

Exercise 4.4 Define fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ as a Gaussian process $(X_t)_{t \in \mathbb{R}_+}$ such $E[X_t] = 0$ for all $t \geq 0$ and the covariance function is given by

$$E[X_t X_s] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

for all $t, s \geq 0$.

We take a continuous version of X and denote it by W^H.

(a) Check that:

- The formula for the covariance is equivalent to the condition
$$E[|X_t - X_s|^2] = |t-s|^{2H}$$
for $t, s \geq 0$, together with $X_0 = 0$ almost surely.
- For $c > 0$, $(\frac{1}{c^H} W^H_{ct})_{t \geq 0}$ is a fBm of Hurst parameter H.
- For $t_0 > 0$, $(W^H_{t+t_0} - W^H_{t_0})_{t \geq 0}$ is a fBm of Hurst parameter H.
- For $H = \frac{1}{2}$, W^H is a Brownian motion.

(b) Use Birkhoff's ergodic theorem to compute the almost sure limit

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{2^n - 1} |W^H_{k+1} - W^H_k|^p$$

for $p > 0$.

(c) Deduce that, for $H < \frac{1}{2}$, W^H has infinite quadratic variation.

(d) **(Python)** Using the scripts available on the lecturer's website, find discrete approximations to the quadratic variation of W^H and convince yourself that (c) holds.