Mathematical Finance

Exercise sheet 7

Exercise 7.1 Consider the Bachelier model, taking $[0,1]$ as the time interval and where the price of the risky asset is given by

$$S_t = \int_0^t \sigma dB_s.$$

Consider $\mathcal{X}_1 := \{\vartheta \bullet S, \vartheta \in \Theta^1_{adm}\}$, the set of wealth processes produced by admissible strategies.

(a) Show that $\mathcal{X}_1$ has the concatenation property: for any bounded, predictable $H, G \geq 0$ with $HG = 0$ and for any $X, Y \in \mathcal{X}_1$, if

$$Z = (H \bullet X) + (G \bullet Y) \geq -1$$

then $Z \in \mathcal{X}_1$.

(b) Show that $\mathcal{X}_1$ is closed in the Emery topology.

Exercise 7.2 Suppose we define a model with time interval $[0,1]$, one riskless asset of constant price 1 and one risky asset which is a compound Poisson process with standard normal jumps.

Specifically, for some Poisson process $(N_t)_{t\in[0,1]}$ of rate 1 and $(Z_k)_{k\in\mathbb{N}}$ a sequence of i.i.d. standard normal variables (also independent from N), we have that

$$S_t = \sum_{k=1}^{N_t} Z_k.$$

We take the natural filtration of S.

Show that the only admissible strategy is 0.

Exercise 7.3 Consider a general model, with $[0,1]$ as the time interval, a riskless asset of constant price 1, and some d-dimensional semimartingale S representing the prices of the risky assets.

Define

$$G = \{(\vartheta \bullet S)_T, \vartheta \in \Theta_{adm}\} \subseteq L^0$$

and

$$C = (G - L^0_{\geq 0}) \cap L^\infty \subseteq L^\infty.$$

(a) Show that the following notions of no arbitrage are equivalent:

$$G \cap L^0_{\geq 0} = \{0\}$$

and

$$C \cap L^\infty_{\geq 0} = \{0\}.$$

(b) Prove that C is weak-$*$-closed if and only for any bounded sequence (f_n) in C converging almost surely to f_0, it holds that $f_0 \in C$.

Exercise 7.4 Suppose that $B = \{\xi_\alpha, \alpha \in A\}$ is some family of non-negative random variables, such that for some $\epsilon > 0$ and all $\alpha \in A$,

$$P(\xi_\alpha \geq \epsilon) \geq \epsilon.$$

Show that

$$0 \notin \overline{\mathrm{co}(B)}^{L^0},$$

where $\mathrm{co}(\cdot)$ is the convex hull.

Exercise 7.5 (Python) Assume Black-Scholes dynamics for S, say $(r, \mu, \sigma) = (0, 0, 1)$, and find the hedging strategy H for the log-contract g whose discounted payoff is given by

$$g(S_T) = \log \frac{S_T}{S_0} + \frac{1}{2}\sigma^2 T.$$

Compare numerically the value of $g(S_T)$ to $(H \bullet S)_T$ at $T = 1$.