Mathematical Finance

Exercise sheet 8

Exercise 8.1 Consider a probability space $(\Omega, \mathcal{F}, P)$, together with a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$. Consider the natural filtration $\mathcal{F}_t^B = \mathcal{F}_t$ generated by B, and suppose that $\mathcal{F}_T = \mathcal{F}$.

(a) Show that any absolutely continuous measure $Q \ll P$ has a Radon-Nikodym derivative of the form

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds\right)$$

for some $\lambda \in L(B)$.

(b) For Q given in the above form, find (with proof) a d-dimensional Brownian motion under Q.

Remark: You may not use Girsanov's theorem for this part!

Exercise 8.2 Consider a discrete time setting with deterministic time points $0 = t_0 < t_1 < t_2 < ... < t_n = T$. In this setting, semimartingales are given in the form

$$S = \sum_{k=0}^{n-1} S_k \mathbb{1}_{[t_k, t_{k+1})} + S_n \mathbb{1}_{\{T\}},$$

where each S_k is $\mathcal{F}_{t_k}$-measurable.

Show that in this case, ucp convergence is equivalent to convergence in Emery topology.

Exercise 8.3 Show that

(a) A local martingale is a sigma-martingale.

(b) A sigma-martingale which is also a special semimartingale is a local martingale.

Exercise 8.4 In the same setup of question 1, consider the Bachelier model:

$$S_t = S_0 + \mu t + \sigma B_t$$

on $[0, T]$, where B is a d-dimensional Brownian motion, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

(a) Show that there exists a unique equivalent martingale measure Q such that for all $f \in L^\infty(\mathcal{F}_T)$, $E_Q(f) = \pi(f)$, where π is the superreplication price.

(b) Take $d = 1$ and $f = (S_T - K)^+$, for some $K \in \mathbb{R}$. Compute $\pi(f)$ as well as the unique strategy ϑ such that

$$\pi(f) + (\vartheta \bullet S)_T = f.$$

(c) Have a look at this paper and write a very short summary of some of the main points.

References

[1] Walter Schachermayer; Josef Teichmann. *How close are the option pricing formulas of Bachelier and Black-Merton-Scholes?* Mathematical Finance, 18: 155-170. doi:10.1111/j.1467-9965.2007.00326.x