Mathematical Finance

Exercise sheet 1

Exercise 1.1 Let $(M_n)_{n\in\mathbb{N}}$ be a martingale such that $M_0 = 0$ and

$$|M_n - M_{n-1}| \leq a_n \quad P\text{-a.s.}$$

for each n and a sequence (a_n) of non-negative constants, with $\sum_{i=1}^{\infty} a_i^2 = A^2 < \infty$.

(a) Prove that M is bounded in L^2. Deduce that $M_n \to M_\infty$ almost surely and in L^2, for some M_∞ in L^2.

(b) Show that

$$P\left(\sup_{k\geq 0} M_k \geq c\right) \leq \exp\left(-\frac{c^2}{2A^2}\right),$$

for any $c > 0$.

Hint: Try applying Doob's maximal inequality to $(e^{\lambda M_n})$, for some $\lambda > 0$. You may use the inequality $\cosh(x) \leq e^{x^2/2}$ (for $x \in \mathbb{R}$).

Solution 1.1

(a) Recall the simple fact that, since M is a martingale,

$$E[(M_{n+1} - M_n)^2 \mid \mathcal{F}_n] = E[M_{n+1}^2 - M_n^2 \mid \mathcal{F}_n].$$

From this and $M_0 = 0$ it follows that

$$\begin{aligned} E[M_n^2] &= \sum_{i=1}^{n} E[(M_n - M_{n-1})^2] \\ &\leq \sum_{i=1}^{n} a_i^2 \leq A^2 < \infty \end{aligned}$$

using the assumptions. Therefore M is bounded in L^2. It is known that boundedness in L^2 implies in particular uniform integrability, so by the martingale convergence theorem there is a limit $M_n \to M_\infty$ almost surely and in L^1. Boundedness of M in L^2 implies furthermore that M_∞ is in L^2 and that the convergence also happens in L^2.

(b) Let $\lambda > 0$ be fixed and let

$$Z_n = e^{\lambda M_n}.$$

Then we have the following (note that, since Z is non-negative, we don't need to assume integrability):

$$E[Z_n \mid \mathcal{F}_{n-1}] = Z_{n-1} E[e^{\lambda(M_n - M_{n-1})} \mid \mathcal{F}_{n-1}].$$

To estimate this term, note that $|M_n - M_{n-1}| \leq a_n$ by assumption. On $[-a_n, a_n]$ we have (by convexity) the inequality

$$\frac{a_n - x}{2a_n} e^{-\lambda a_n} + \frac{a_n + x}{2a_n} e^{\lambda a_n} \geq e^{\lambda x}$$

(this simply follows from convexity). Thus,

$$\begin{aligned}
E[Z_n \mid \mathcal{F}_{n-1}] &\leq Z_{n-1} \left(\frac{a_n - E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}]}{2a_n} e^{-\lambda a_n} + \frac{a_n + E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}]}{2a_n} e^{\lambda a_n} \right) \\
&= Z_{n-1} \left(\frac{1}{2} e^{-\lambda a_n} + \frac{1}{2} e^{\lambda a_n} \right) \\
&= Z_{n-1} \cosh(\lambda a_n) \\
&\leq Z_{n-1} e^{\lambda^2 a_n^2/2},
\end{aligned}$$

using that M is a martingale and the given inequality.

Iterating, we obtain

$$E[Z_n] \leq \exp\left(\lambda^2 \sum_{i=1}^{n} a_i^2/2 \right) \leq \exp(\lambda^2 A^2/2).$$

In particular, this proves that Z_n is integrable, and from Jensen's inequality it follows easily that Z is a submartingale (since M is a martingale).

Next, we apply Doob's maximal inequality to Z to obtain the following (let $M_n^* = \max_{0 \leq k \leq n} M_k$, etc):

$$\begin{aligned}
P(M_n^* \geq c) &= P(Z_n^* \geq e^{\lambda c}) \\
&\leq e^{-\lambda c} E[Z_n] \\
&\leq e^{-\lambda c + \lambda^2 A^2/2}.
\end{aligned}$$

At this point we haven't specified what value $\lambda > 0$ will take, and so we are free to choose a convenient one. We choose λ so as to minimise the exponent, meaning that $\lambda = \frac{c}{A^2}$ and so

$$P(M_n^* \geq c) \leq \exp\left(-\frac{c^2}{2A^2} \right),$$

which is precisely the bound we want. To replace M_n^* with M_∞^* we simply use the monotone convergence theorem.

Exercise 1.2 Let $\mathbb{S}$ denote the family of simple predictable processes H, i.e.

$$H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}$$

for stopping times $0 = \tau_0 < \tau_1 < ... < \tau_{i+1} < \infty$ and bounded $\mathcal{F}_{\tau_i}$-measurable H_i for $i = 0, 1, ..., n+1$. Let $\mathbb{D}$ denote the family of adapted càdlàg processes and $\mathbb{L}$ denote the family of adapted càglàd processes on $[0, \infty)$. We endow $\mathbb{D}$ and $\mathbb{L}$ with the topology of convergence uniformly on compacts in probability, generated by the metric

$$d(X, Y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \mathbb{E}[|(X - Y)|_k^* \wedge 1].$$

Moreover, let the space of all measurable random variables L^0 be endowed with the topology generated by convergence in probability. Show that:

(a) The vector spaces $\mathbb{L}$ and $\mathbb{D}$ are complete.

(b) For some càdlàg process X, the following are equivalent:

1. The map $J_X : \mathbb{S} \to \mathbb{D}$ with $J_X(H) := H_0 X_0 + \sum_{i=1}^n H_i (X_{\tau_{i+1}\wedge\cdot} - X_{\tau_i \wedge \cdot})$, for $H \in \mathbb{S}$, is continuous with respect to the u.c.p. metric on $\mathbb{S}$ and $\mathbb{D}$, in other words, X is a good integrator.
2. For every $t \in [0, \infty)$, the mapping $I_{X^t} : \mathbb{S} \to L^0$ with $I_{X^t}(H) := J_X(H)_t$, for $H \in \mathbb{S}$, is continuous with respect to the uniform norm metric on $\mathbb{S}$.

Solution 1.2

(a) Let $(X^n)_{n\in\mathbb{N}}$ be a Cauchy sequence (with respect to ucp metric) in $\mathbb{L}$ or $\mathbb{D}$. In other words, $d(X^m, X^n) \to 0$ as $m, n \to 0$. We may extract a subsequence $Y^k := X^{n_k}$ such that

$$d(Y^m, Y^n) = \sum_{k=1}^{\infty} \frac{1}{2^k} E\left[|(Y^m - Y^n)|_k^* \wedge 1\right] \leq 2^{-n}$$

for all $m \geq n$.

Then (since the summands are non-negative), we have

$$E\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} |(Y^{n+1} - Y^n)|_k^* \wedge 1\right] = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} E\left[|(Y^{n+1} - Y^n)|_k^* \wedge 1\right] \leq 1.$$

We conclude from this that the sum inside the expectation on the left is finite almost surely. This implies that, for $\omega \in A$ where A is a subset of full measure, we have that $(Y^n(\omega))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_{\text{loc}}(\mathbb{R}_+)$. Since this space is complete, there is a limit $Y^n(\omega) \to X(\omega)$ in $L^\infty_{\text{loc}}(\mathbb{R}_+)$ for $\omega \in A$. This is our candidate limit for X (defined arbitrarily outside of A). The almost sure convergence of (Y^n) implies the (weaker) convergence of (Y^n) in ucp topology. Since (X^n) is a Cauchy sequence with respect to the ucp metric, it follows that the whole sequence (X^n) converges to X in ucp topology.

The limit is clearly measurable, and moreover

$$X_t = \lim_{n\to\infty} Y_t^n$$

for every $t \in \mathbb{R}_+$, so that X is adapted. The almost sure uniform convergence on compacts of (Y^n) to X gives that $X \in \mathbb{L}$ or $\mathbb{D}$ if X^n (and hence Y^n) are in the corresponding space.

(b) Since J_X and $I_{X^t}, t \in [0, \infty)$ are linear mappings between topological vector spaces, it suffices to establish the continuity at the origin of $\mathbb{S}$. Let X be a good integrator and $H^n \to 0$ uniformly. This implies the weaker convergence $H^n \to 0$ in ucp topology. Since X is a good integrator, $J_X(H^n) \to 0$ in ucp topology. In particular, for any $t \in [0, \infty)$ we have $I_{X^t}(H^n) \to 0$ in probability, as claimed.

For the converse, let $H^n \to 0$ in ucp topology, and take $c > 0$. Fix a $t \geq 0$ and $\epsilon > 0$. By assumption, we can find a $\delta > 0$ such that if $||H||_\infty \leq \delta$, then $P(|H \bullet X|_t > c) < \epsilon$.

Now, for each $n \in \mathbb{N}$ define the stopping times

$$\begin{aligned}\tau^n &= \inf\{s \geq 0 : H_s^n > \delta\},\\ \sigma^n &= \inf\{s \geq 0 : |(H^n \mathbb{1}_{[0,\tau^n]} \bullet X)|_s > c\}.\end{aligned}$$

Then, the following inequalities hold:

$$\begin{aligned} P(|H^n \bullet X|_t^* > c) &\le P(|H^n \mathbb{1}_{[0,\tau^n]} \bullet X|_t^* > c) + P(\tau^n \le t) \\ &\le P(|H^n \mathbb{1}_{[0,\tau^n \wedge \sigma^n]} \bullet X|_t > c) + P(\tau^n \le t) \\ &\le \epsilon + P(\tau^n \le t) \\ &\le 2\epsilon \end{aligned}$$

for large enough n. The first line follows, since $H^n = H^n \mathbb{1}_{[0,\tau^n]}$ up to time t if $\{\tau^n > t\}$ holds. The second line is justified by the fact that $\{|H^n \mathbb{1}_{[0,\tau^n]} \bullet X|_t^* > c\}$ and $\{|H^n \mathbb{1}_{[0,\tau^n \wedge \sigma^n]} \bullet X|_t > c\}$ are both the same event as $\{\sigma^n \le t\}$. For the third line, note that $||H^n \mathbb{1}_{[0,\tau^n]}||_\infty \le \delta$ by definition of τ_n and left-continuity of H^n, and therefore we can use the assumption on I_t to bound this probability. For the last line, note that $\{\tau^n \le t\} = \{\sup_{s \in [0,t]} |H_s^n| > c\}$ has low probability for large n, thanks to ucp convergence.

Putting everything together, we see that $H^n \bullet X \to 0$ in ucp topology, which proves that X is a good integrator.

Exercise 1.3 Prove that the set of good integrators is a vector space and show that it is generically not closed with respect to the ucp topology. Construct in particular examples of processes which are not good integrators but can be approximated by good integrators.

Solution 1.3 To prove that the set of good integrators is a vector space, we simply use the linearity of stochastic integrals. Let $\lambda \in \mathbb{R}$, take X, Y good integrators and H^k a sequence of simple integrands with $H^k \to 0$ in ucp topology. By linearity of stochastic integration (which is trivial to check for simple integrands),

$$H^k \bullet (X + \lambda Y) = H^k \bullet X + \lambda H^k \bullet Y.$$

Since $H^k \to 0$ in ucp topology, each of these terms converges to 0 in ucp topology (as X, Y are good integrators). Since $\mathbb{D}$ with ucp topology is a topological vector space, $H^k \bullet (X + \lambda Y)$ also converges to 0 in ucp topology.

To prove that the set of good integrators is not closed with respect to the ucp topology, we can even construct a simple deterministic example. Let $0 = t_0 < t_1 < t_2 < ... < 1$ be an increasing sequence converging to 1. Let $(a_n)_{n \ge 0}$ be a non-negative sequence converging to 0 with $\sum a_n = \infty$ (e.g. $a_n = \frac{1}{n}$). Now define the following process on $[0,1]$ which alternates between 0 and (a_n):

$$X_t = \sum_{k=0}^{\infty} a_k \mathbb{1}_{t \in (t_{2k}, t_{2k+1}]}.$$

We define furthermore

$$X_t^k = X_t \mathbb{1}_{t \in [t_0, t_{2k}]}.$$

It is clear that $X^k \to X$ uniformly on $[0,1]$: indeed,

$$\sup_{t \in [0,1]} |X_t^k - X_t| = \sup_{j \ge k+1} a_j \to 0$$

since $a_j \to 0$.

Moreover, each X^k is a good integrator: since X^k only has finitely many jumps, it is of finite variation and thus a good integrator (from the lecture notes).

Finally, X is not a good integrator. To see this, take the sequence of midpoints $s_n = \frac{t_{n-1}+t_n}{2}$, so that $t_0 < s_1 < t_1 < s_2 < ... < 1$. Consider

$$H_t = \sum_{k=1}^{\infty} \mathbb{1}_{t \in [s_{2k-1}, s_{2k})}$$

(which is not a simple integrand, since it has infinitely many jumps), and given a non-negative sequence $\lambda_n \to 0$ to be chosen later, the simple integrands

$$H_t^n = \lambda_n H_t \mathbb{1}_{t\in[0,t_{2n}]}.$$

Since $||H||_\infty = 1$, it is clear that $H^n \to 0$ uniformly. Moreover,

$$\begin{aligned}(H^n \bullet X)_1 &= \lambda_n \sum_{k=1}^{n} (X_{s_{2k}} - X_{s_{2k-1}}) \\ &= \lambda_n \sum_{k=1}^{n} (-a_{k-1}) \\ &= -\lambda_n \sum_{k=0}^{n-1} a_k.\end{aligned}$$

By choosing an appropriate sequence λ_n, say $\lambda_n = \left(\sum_{k=0}^{n-1} a_k\right)^{-\frac{1}{2}}$, this will diverge, therefore proving that X is not a good integrator.

Exercise 1.4 Let μ be a probability measure on $(0, +\infty)$. Consider (on some probability space) independent $N, Y_1, Y_2, Y_3, ...$ where each Y_i has distribution μ and $N = (N_t)_{t\in[0,1]}$ is a Poisson process on $[0, 1]$ of rate $\lambda > 0$. Consider the compound Poisson process X on $[0, 1]$ given by

$$X_t = \sum_{i=1}^{N_t} Y_i.$$

(a) Find a necessary and sufficient condition for X to be a submartingale with respect to its natural filtration.

(b) Show that under that condition, X is a submartingale of class (D). Find a decomposition

$$X_t = M_t + A_t \quad \forall t \in [0, 1],$$

where M is a martingale and A is an increasing predictable process, both with càdlàg trajectories.

(c) Show through direct calculations that X is a good integrator.

Solution 1.4

(a) Since the Y_i are non-negative, the following computations hold for $t \in [0, 1]$:

$$
\begin{aligned}
E[X_t] &= E\left[\sum_{i=1}^{N_t} Y_i\right] \\
&= E\left[E\left[\sum_{i=1}^{N_t} Y_i \mid N_t\right]\right] \\
&= E\left[\sum_{i=1}^{N_t} E\left[Y_i \mid N_t\right]\right] \\
&= E\left[N_t \int_{(0,\infty)} x d\mu(x)\right] \\
&= \lambda t \int_{(0,\infty)} x d\mu(x),
\end{aligned}
$$

using independence and the distribution of the Y_i.

Therefore, if X is going to be a submartingale then $\int_{(0,\infty)} x d\mu(x) =: \mu_0 < \infty$ is required. This is in fact sufficient: the calculations above show that X is integrable. It is obviously adapted to its natural filtration, and since it is (almost surely) increasing it must be a submartingale.

(b) Note that since X is increasing, for any stopping time τ (taking values on $[0,1]$), $X_\tau \leq X_1$. Since X_1 is integrable (and X_τ is non-negative), this means that $\{X_\tau : \tau \text{ is a stopping time on } [0,1]\}$ is uniformly integrable. Therefore $(X_t)_{t\in[0,1]}$ is of class (D).

We find the required decomposition:

$$
X_t = (X_t - \lambda\mu_0 t) + \lambda\mu_0 t.
$$

Clearly this is a valid decomposition, and $A_t = \lambda\mu_0 t$ is increasing, predictable (even deterministic) and càdlàg. $M_t = X_t - \lambda\mu_0 t$ is clearly càdlàg since X is, and we want to show that it is a martingale.

Since X is adapted and integrable, so is M. To show that it is a martingale, note that

$$
\begin{aligned}
E[M_t \mid F_s] &= M_s - \lambda(t-s) + E\left[\sum_{i=N_s+1}^{N_t} Y_i \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[E\left[\sum_{i=N_s+1}^{N_t} Y_i \mid \sigma(N_t, \mathcal{F}_s)\right] \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[\sum_{i=N_s+1}^{N_t} E\left[Y_i \mid \sigma(N_t, \mathcal{F}_s)\right] \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[\sum_{i=N_s+1}^{N_t} \mu_0 \mid \mathcal{F}_s\right] \\
&= M_s - \lambda\mu_0(t-s) + E\left[\mu_0(N_t - N_s) \mid \mathcal{F}_s\right] \\
&= M_s,
\end{aligned}
$$

as we wanted (using again independence, the distribution of the Y_i as well as independence of increments of the Poisson process).

(c) Given a simple integrand $H = H_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} H_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}$, we have that

$$\begin{aligned}
|(H \bullet X)_t| &= \left| \sum_{i=1}^{n} H_i (X_{\tau_{i+1} \wedge t} - X_{\tau_i \wedge t}) \right| \\
&\leq \sum_{i=1}^{n} |H_i| |X_{\tau_{i+1} \wedge t} - X_{\tau_i \wedge t}| \\
&\leq \sum_{i=1}^{n} |H_i| (X_{\tau_{i+1}} - X_{\tau_i}) \\
&\leq X_1 \sup_{t \in [0,1]} |H_t|
\end{aligned}$$

Now, if we take simple integrands H^k converging to 0 in ucp topology, then we have that for $\epsilon > 0$,

$$\begin{aligned}
P(\sup_{t \in [0,1]} |(H^k \bullet X)_t| > \epsilon) &\leq P(\sup_{t \in [0,1]} |H_t^k| X_1 > \epsilon) \\
&\leq P\left(\sup_{t \in [0,1]} |H_t^k| > \frac{\epsilon}{M} \right) + P(X_1 > \epsilon M)
\end{aligned}$$

for any $M > 0$. By choosing M large enough we can make the second term small, and then by choosing k large enough we can make the first term small as well (using ucp convergence of H^k to 0). This shows convergence of $H^k \bullet X$ to 0 in the ucp topology.