Mathematical Finance

Exercise sheet 11

Exercise 11.1 Let U be a utility function satisfying the Inada conditions, i.e. $U \in C^1(\mathbb{R}_+;\mathbb{R})$ is strictly increasing, strictly concave and

$$U'(0) := \lim_{x \searrow 0} U'(x) = +\infty$$
$$U'(+\infty) := \lim_{x \to +\infty} U'(x) = 0.$$

Let J be the Legendre transform of $-U(-\cdot)$,

$$J(y) := \sup_{x>0}(U(x) - xy),$$

and denote by $I := (U')^{-1}$ the inverse of the derivative of U.

Show the following properties:

1. J is strictly decreasing and strictly convex.
2. $J'(0) = -\infty$, $J'(+\infty) = 0$, $J(0) = U(+\infty)$ and $J(+\infty) = U(0)$.
3. For any $x > 0$,
$$U(x) = \inf_{y>0}(J(y) + xy).$$
4. For any $y > 0$,
$$J(y) = U(I(y)) - yI(y).$$
5. $J' = -I$.

Solution 11.1

First we show that the supremum defining J is a maximum, i.e. for any $y > 0$, we have

$$J(y) = \sup_{x>0}(U(x) - xy) = U(x_y) - x_y y$$

for some $x_y > 0$.

Note that, letting $g_y(x) = U(x) - xy$, we have that g_y is differentiable and

$$g'_y(x) = U'(x) - y.$$

Since U is C^1 (i.e. U' is continuous), strictly convex (i.e. U' is strictly decreasing), with $U'(0) = +\infty$ and $U'(+\infty) = 0$ by the Inada conditions, we obtain exactly one solution to $U'(x_y) = y$. Moreover, g'_y is negative for $x > x_y$ and positive for $x < x_y$. Thus the maximum is obtained exactly at $x_y = (U')^{-1}(y)$, which is a continuous, decreasing function of y (since U' is). We keep using the notation x_y.

On the other hand, we can see that for $y \leq 0$, the maximum is obtained as $x \to +\infty$ (since g_y is increasing in x), which gives that $J(0) = U(+\infty)$ (possibly $= +\infty$) and $J(y) = +\infty$ for $y < 0$.

We show first that J is differentiable on $(0, +\infty)$. Note that the equation $U'(x_y) = y$ implies that x_y is increasing in y. Thus, we have the following: picking some arbitrary $\bar{y}$, and letting $\bar{x} = x_{\bar{y}}$,

$$\begin{aligned} J(y) &= U(x_y) - x_y y \\ &= U(\bar{x}) - \bar{x}\bar{y} + \int_{\bar{x}}^{x_y} U'(w)dw - \int_{\bar{y}}^{y} (sdx_s + x_s ds) \\ &= U(\bar{x}) - \bar{x}\bar{y} + \int_{\bar{y}}^{y} U'(x_s)dx_s - \int_{\bar{y}}^{y} (sdx_s + x_s ds) \end{aligned}$$

(using a Riemann-Stieltjes integral, chain rule and integration by parts).

Since $U'(x_s) = s$, this simplifies as

$$J(y) = J(\bar{y}) - \int_{\bar{y}}^{y} x_s ds.$$

But since $x_s = (U')^{-1}(s)$ is a continuous function of s, this shows that J is differentiable with $J'(y) = -x_y$.

1. J is strictly decreasing and strictly convex since $J'(y) = -(U')^{-1}(y)$ is a strictly negative and strictly increasing function of y.

2. We have $J'(0) = -(U')^{-1}(0) = -\infty$ and $J'(+\infty) = -(U')^{-1}(+\infty) = 0$, by the Inada conditions. We already saw earlier that $J(0) = U(+\infty)$.

   Note that $J(y) = \sup_{x>0}(U(x) - xy) \geq U(0)$ for any $y > 0$, by taking $x \to 0$. Moreover, $J(y) \leq U(\epsilon)$ for small enough $y > 0$, since for $y \geq U'(\epsilon)$, we have

   $$U(x) - xy \leq U(x) \leq U(\epsilon)$$

   if $x \leq \epsilon$, and we know that the maximiser x_y must be in $[0, \epsilon]$ (since U' is decreasing). Thus, taking $\epsilon \to 0$ we get $J(+\infty) = U(0)$.

3. By definition, $J(y) = \sup_{x>0}(U(x) - xy)$, from which we see that

   $$J(y) - U(x) \geq -xy$$

   for any $y, x \geq 0$. We can also write it as

   $$U(x) \leq J(y) + xy.$$

   On the other hand, for any $x > 0$ we know that this inequality is attained at $y = U'(x)$, and therefore

   $$U(x) = \inf_{y>0} J(y) + xy.$$

   In the case of $x = 0$, we saw before that $U(0) = J(+\infty)$, and we can show that this is equal to the infimum.

4. We already showed this above.

5. Likewise.

Exercise 11.2 Let the financial market $S = (S_k)_{k=0,\ldots,N}$ be defined over the *finite* filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,\ldots,N}, P)$ and satisfy $\mathcal{M}^a(S) \neq \emptyset$, and let U be a utility function satisfying the Inada conditions. Consider the value functions

$$u(x) = \sup_{X_T \in C(x)} E[U(X_T)] \text{ and } v(y) = \inf_{Q \in \mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right],$$

where V is the convex conjugate of U and

$$C(x) = \{X_T \in L^0(\Omega, \mathcal{F}_T, P) \mid \forall Q \in \mathcal{M}^a(S) : E_Q[X_T] \leq x\}.$$

Show that the optimisers $\hat{X}_T(x)$, $\hat{Q}(x)$ and $\hat{y}(x)$ satisfy $U'\left(\hat{X}_T(x)\right) = \hat{y}(x)\frac{d\hat{Q}(x)}{dP}$ for each $x \in \mathrm{dom}(U)$.

Solution 11.2 From minimax considerations, and writing the Lagrangian $L(X_T, y, Q) = E\left[U(X_T) - y\left(\frac{dQ}{dP}X_T - x\right)\right]$ for $X_T \in L^0(\Omega)$, $y > 0$ and $Q \in \mathcal{M}^a(S)$, we have

$$\begin{aligned}
\sup_{X_T \in C(x)} E[U(X_T)] &= \sup_{X_T} \inf_{y>0, Q\in\mathcal{M}^a(S)} L(X_T, y, Q) \\
&= \sup_{X_T} \inf_{y>0, Q\in\mathcal{M}^a(S)} E\left[U(X_T) - y\left(\frac{dQ}{dP}X_T - x\right)\right] \\
&= \inf_{y>0, Q\in\mathcal{M}^a(S)} \sup_{X_T} E\left[U(X_T) - y\left(\frac{dQ}{dP}X_T - x\right)\right] \\
&= \inf_{y>0, Q\in\mathcal{M}^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right] - xy \\
&= \inf_{y>0, Q\in\mathcal{M}^a(S)} v(y) - xy.
\end{aligned}$$

Note that we obtain the third line by maximising for each fixed ω, since we no longer have any constraint on X_T (other than measurability). From question 1 we know that this supremum is attained exactly when $U'(X_T) = y\frac{dQ}{dP}$. Furthermore, the last infimum is attained exactly when $y = \hat{y}(x)$, and the infimum over the martingale measure is obtained exactly when $\frac{dQ}{dP} = \frac{d\hat{Q}(x)}{dP}$. Likewise, the supremum over X_T is attained exactly when $\hat{X}_T(x)$. Therefore, the unique saddle point of the Lagrangian satisfies $U'\left(\hat{X}_T(x)\right) = \hat{y}(x)\frac{d\hat{Q}(x)}{dP}$, as we wanted.

Exercise 11.3 Let $C \subseteq L^0_+$ be convex, closed and bounded in probability, and assume that $1 \in C$. Let $D := \{z \in L^0_+ \mid \forall f \in C : \, E[zf] \leq 1\}$. Show that for any $g \in L^\infty$,

$$\inf\{x \in \mathbb{R}_+ \mid \exists f \in C : \, xf \geq g\} = \sup_{z\in D} E[zg].$$

Solution 11.3 The inequality $\geq$ is clear, since if $xf \geq g$, then $x \geq xE[zf] \geq E[zg]$ for any $z \in D$, and we can take infimum and supremum on each side of the inequality.

For the other inequality, let $x_0 = \sup_{z\in D} E[zg]$. Let $K = \overline{(C - L^1) \cap L^1}^{L^1}$. One can see that as in Kardaras that K is convex and closed in L^1. Suppose that $\frac{g}{x_0} \notin K$. Then, by the Hahn-Banach separation theorem, we can find some $z \in L^\infty$ such that $E[zf] \leq 1$ for all $f \in C$ and $E\left[z\frac{g}{x_0}\right] > 1$. Since $E[zf] \leq 1$ for any $f \leq 0$, we easily see that $z \geq 0$ a.s.. Then, $z \in D$. But $E\left[z\frac{g}{x_0}\right] = \frac{E[zg]}{\sup_{z\in D} E[zg]} \leq 1$, leading to a contradiction.

Therefore, $\frac{g}{x_0} \in K$, so that we can find $f \in C$ with $f \geq \frac{g}{x_0}$, i.e. $x_0 f \geq g$, as we wanted.