Mathematical Finance

Exercise sheet 2

Exercise 2.1 Let W^1, W^2 be two independent Brownian motions.

(a) For two C^2 functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, let $X_t = f(W_t^1, W_t^2)$ and $Y_t = g(W_t^1, W_t^2)$. Compute the quadratic covariation of X and Y.

(b) Let $Z_t = \int_0^t \left(\cos s dW_s^1 + \sin s dW_s^2\right)$. Prove that Z is a Brownian motion.

Solution 2.1

(a) By Itô's formula, and using that $[W^1]_t = [W^2]_t = t$ and $[W^1, W^2]_t = 0$, as well as noting that, by continuity, we may take time s instead of $s-$ in the integrands, we obtain the following:

$$\begin{aligned} X_t = f(W_t^1, W_t^2) = f(0,0) &+ \int_0^t f_x(W_s^1, W_s^2) dW_s^1 + \int_0^t f_y(W_s^1, W_s^2) dW_s^2 \\ &+ \frac{1}{2}\left(\int_0^t f_{xx}(W_s^1, W_s^2) ds + \int_0^t f_{yy}(W_s^1, W_s^2) ds\right), \end{aligned}$$

where f_x is the partial derivative with respect to the first variable, etc.

We can also write in integral notation

$$X = f(0,0) + f_x(W_\cdot^1, W_\cdot^2) \bullet W^1 + f_y(W_\cdot^1, W_\cdot^2) \bullet W^2 + \frac{1}{2}(f_{xx}(W_\cdot^1, W_\cdot^2) + f_{yy}(W_\cdot^1, W_\cdot^2)) \cdot I,$$

where $I_t = t$. Alternatively, we can write in differential notation:

$$dX_t = f_x(W_t^1, W_t^2) dW_t^1 + f_y(W_t^1, W_t^2) dW_t^2 + \frac{1}{2}(f_{xx}(W_t^1, W_t^2) + f_{yy}(W_t^1, W_t^2)) dt.$$

Similarly, Itô's formula applied to Y yields

$$Y = g(0,0) + g_x(W_\cdot^1, W_\cdot^2) \bullet W^1 + g_y(W_\cdot^1, W_\cdot^2) \bullet W^2 + \frac{1}{2}(g_{xx}(W_\cdot^1, W_\cdot^2) + g_{yy}(W_\cdot^1, W_\cdot^2)) \cdot I.$$

Now, we can use the properties of stochastic integration and of quadratic variation to compute the quadratic covariation of these two processes. Note that the constant terms and the integrals against I do not matter in computing the quadratic covariation, since they are continuous and of finite variation (and therefore of null quadratic variation). We are left with the integrals against the two Brownian motions. By Itô's isometry (Proposition 6.7 in the lecture notes), as well as associativity (Proposition 5.5), we obtain

$$\begin{aligned} [f_x(W_\cdot^1, W_\cdot^2) \bullet W^1, g_x(W_\cdot^1, W_\cdot^2) \bullet W^1] &= f_x(W_\cdot^1, W_\cdot^2) \cdot [W^1, g_x(W_\cdot^1, W_\cdot^2) \bullet W^1] \\ &= f_x(W_\cdot^1, W_\cdot^2) \cdot (g_x(W_\cdot^1, W_\cdot^2) \cdot [W^1, W^1]) \\ &= f_x(W_\cdot^1, W_\cdot^2) \cdot (g_x(W_\cdot^1, W_\cdot^2) \cdot I) \\ &= (f_x(W_\cdot^1, W_\cdot^2) g_x(W_\cdot^1, W_\cdot^2)) \cdot I. \end{aligned}$$

We can perform similar computations for the other pairs (note that $[W^1, W^2] = 0$, so the cross-terms vanish). Therefore, we finally obtain

$$[X,Y]_t = \int_0^t f_x(W_s^1, W_s^2) g_x(W_s^1, W_s^2) ds + \int_0^t f_y(W_s^1, W_s^2) g_y(W_s^1, W_s^2) ds.$$

(b) It is easy to see that, since cos and sin are bounded, Z is a local martingale. In more detail, note that for each $T > 0$,

$$Z_{t\wedge T} = \int_0^t \left(\cos s dW^1_{s\wedge T} + \sin s dW^2_{s\wedge T}\right)$$

is a martingale in $\mathcal{H}^1$ (e.g. by Proposition 6.7). Since this holds for any deterministic $T > 0$, Z itself is a martingale (and in particular a local martingale).

Next, we compute the quadratic variation. Once again using Itô's isometry and associativity, along with the standard quadratic covariations of the Brownian motions, we obtain e.g.

$$\begin{aligned}[\cos(\cdot) \bullet W^1] &= [\cos(\cdot) \bullet W^1, \cos(\cdot) \bullet W^1] \\ &= \cos(\cdot) \cdot (\cos(\cdot) \cdot [W^1, W^1]) \\ &= \cos(\cdot)^2 \cdot I,\end{aligned}$$

so that we obtain

$$[Z]_t = \int_0^t (\cos(s)^2 + \sin(s)^2) ds = t.$$

Noting that Z is a local martingale starting at 0 with the correct quadratic variation, Lévy's criterion gives that Z is a Brownian motion, as we wanted.

Exercise 2.2 Let $N^1, ..., N^m$ and $W^1, ..., W^m$ all be independent, with each N^k a Poisson process of rate 1 and each W^k a Brownian motion, all starting at 0. Let $X^k = N^k + W^k$ for each k.

(a) Recall the formula for the stochastic exponential of a process given in the lecture notes. Find the stochastic exponential Z of $X = \sum_{k=1}^m X^k$, and check directly that it satisfies the stochastic differential equation (SDE)

$$dZ = Z_- dX, \quad Z_0 = 1.$$

By this we mean that the integrated form of this equation holds:

$$Z_t - 1 = (Z_- \bullet X)_t.$$

(b) Use Itô's formula to find a decomposition for the process

$$Y_t = |\mathbf{X}_t|^{2\alpha},$$

where $\mathbf{X}_t = (X_t^1, ..., X_t^m)$ and $|\mathbf{X}_t| = \left(\sum_{k=1}^m (X_t^k)^2\right)^{\frac{1}{2}}$, and we also assume $\alpha \in \mathbb{N}$.

(c) (optional) Let $\mathbf{v} \in \mathbb{R}^m$ and suppose that $P(\forall t \geq 0 \ \mathbf{X}_t, \mathbf{X}_{t-} \neq \mathbf{v}) = 1$.

Find a similar decomposition for the process

$$\tilde{Y}_t = |\mathbf{X}_t - \mathbf{v}|^{2\alpha},$$

where now $\alpha \in \mathbb{R}$.

Solution 2.2 We use the simplified version of Itô's lemma explained in class, i.e. if $X = X^c + X^j$ is a semimartingale decomposition with X^c continuous and X^j a pure jump process, then for f a C^2 function,

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s^c + \frac{1}{2}\int_0^t f''(X_{s-})d[X^c]_s + \sum_{s\le t}(f(X_s) - f(X_{s-})).$$

It can similarly be generalised to higher-dimensional processes.

(a) We likewise use the simplified formula for the stochastic exponential when such a decomposition is available. The stochastic exponential is then given by

$$Z_t = \exp(X_t^c - \frac{1}{2}[X^c]_t)\prod_{s\le t}(1 + \Delta X_s^j).$$

For our given X, note that the quadratic variation of the continuous part (made up of Brownian motions) can be readily computed, since we know all the covariations. Note also that the jump part, as the sum of independent Poisson processes, is itself a Poisson process of rate m, and therefore (almost surely) all of its jumps are of size 1. Therefore each of the contributions to the product is a factor of size 2. We obtain the exponential as

$$Z_t = \exp\left(\sum_{k=1}^m W_t^k - \frac{mt}{2}\right) 2^{\sum_{k=1}^m N_t^k}.$$

To check that this truly is the stochastic exponential, note that it clearly is equal to 1 at time 0, and moreover, by (the simplified) Itô's formula, as well as some simplifications using the covariations of the Brownian motions, we obtain

$$Z_t = 1 + \sum_{k=1}^m \int_0^t Z_{s-}dW_s^k - \int_0^t \frac{m}{2}Z_{s-}ds + \frac{1}{2}\left(m\int_0^t Z_{s-}ds\right) + \sum_{s\le t}(Z_s - Z_{s-}).$$

The middle two terms cancel. Note that, whenever a jump occurs, Z is multiplied by 2. Therefore, each jump $Z_s - Z_{s-}$ is the same as Z_{s-}. Moreover, note that the jumps occur whenever $X^j = \sum_{k=1}^m N^k$ jumps, and X^j has jumps of size 1. Therefore, we can further write

$$\sum_{s\le t}(Z_s - Z_{s-}) = \sum_{s\le t} Z_{s-} = (Z_- \cdot X^j)_t.$$

This finally gives that

$$Z = 1 + (Z_- \bullet X^c) + (Z_- \cdot X^j) = 1 + Z_- \bullet X,$$

as required.

(b) Similarly, the simplified version of Itô's formula applied to $f(\mathbf{x}) = |\mathbf{x}|^{2\alpha}$ (which is a smooth function for $\alpha \in \mathbb{N}$) gives

$$Y_t = \sum_{k=1}^m \int_0^t 2\alpha Y_{s-}^{\frac{\alpha-1}{\alpha}}(W_s^k + N_{s-}^k)dW_s^k + \frac{1}{2}\sum_{k=1}^m\int_0^t \left(2\alpha Y_{s-}^{\frac{\alpha-1}{\alpha}} + 4\alpha(\alpha-1)Y_{s-}^{\frac{\alpha-2}{\alpha}}(W_s^k + N_{s-}^k)^2\right)ds + \sum_{s\le t}(Y_s - Y_{s-}).$$

By summing up the middle terms, and noting that $\sum_{k=1}^m (W_s^k + N_{s-}^k)^2 = Y_{s-}^{\frac{1}{\alpha}}$, we obtain

$$Y_t = \sum_{k=1}^m \int_0^t 2\alpha Y_{s-}^{\frac{\alpha-1}{\alpha}} (W_s^k + N_{s-}^k) dW_s^k + \alpha(m + 2(\alpha - 1)) \int_0^t Y_{s-}^{\frac{\alpha-1}{\alpha}} ds + \sum_{s \le t} (Y_s - Y_{s-}).$$

This formula holds for all $\alpha \in \mathbb{N}$ except $\alpha = 0$, in which case $Y_t = 1$ is trivial.

(c) By applying Itô's formula similarly to the previous part, we obtain essentially the same result for $\alpha \in \mathbb{N}$:

$$Y_t = Y_0 + \sum_{k=1}^m \int_0^t 2\alpha Y_{s-}^{\frac{\alpha-1}{\alpha}} (W_s^k + N_{s-}^k) dW_s^k + \alpha(m + 2(\alpha - 1)) \int_0^t Y_{s-}^{\frac{\alpha-1}{\alpha}} ds + \sum_{s \le t} (Y_s - Y_{s-}).$$

(note the now non-trivial initial condition $Y_0 = |\mathbf{v}|^{2\alpha}|$).

If we want to extend this for $\alpha \in \mathbb{R}$, Itô's formula alone does not suffice, since $f(\mathbf{x}) = |\mathbf{x} - \mathbf{v}|^{2\alpha}$ is not a C^2 function in general (or even well-defined at $\mathbf{x} = \mathbf{v}$).

Instead, consider $\tau_n = \inf\{t > 0 : |\mathbf{X}_t - \mathbf{v}| < \frac{1}{n}\}$ and a sequence of C^2 functions f_n such that $f_n(\mathbf{x}) = f(\mathbf{x})$ whenever $|\mathbf{x} - \mathbf{v}| \ge \frac{1}{n}$. Such functions are easy to construct in this case: one possible approach is to find the correct constants $a_n, b_n, c_n \in \mathbb{R}$ such that the function

$$\tilde{f}_n(r) = \begin{cases} |r|^{2\alpha}, & |r| \ge \frac{1}{n} \\ a_n e^{-b_n x^2} + c_n, & |r| < \frac{1}{n} \end{cases}$$

is C^2 on all of $\mathbb{R}$, and then to take $f_n(\mathbf{x}) = \tilde{f}_n(|\mathbf{x} - \mathbf{v}|)$ (this is best understood by drawing a graph). It is an easy check to see that this can be done, and that the resulting functions are C^2.

Now we are equipped with f_n and τ_n. By Itô's theorem, since the f_n are C^2, the corresponding Itô formulas hold for each f_n. Moreover, note that, by definition of the τ_n, $|\mathbf{X}_t - \mathbf{v}| \ge \frac{1}{n}$ for all t in the stochastic interval $[0, \tau_n)$. Since f_n and f coincide whenever $|\mathbf{x} - \mathbf{v}| \ge \frac{1}{n}$, it follows that, for each n,

$$Y_t = Y_0 + \sum_{k=1}^m \int_0^t 2\alpha Y_{s-}^{\frac{\alpha-1}{\alpha}} (W_s^k + N_{s-}^k) dW_s^k + \alpha(m + 2(\alpha - 1)) \int_0^t Y_{s-}^{\frac{\alpha-1}{\alpha}} ds + \sum_{s \le t} (Y_s - Y_{s-}).$$

holds (almost surely) on $[0, \tau_n)$.

By countability, it follows that the same equation holds almost surely on the union of those intervals, $[0, \lim_{n\to\infty} \tau_n)$ (since the sequence τ_n is increasing). All that remains is to show that this limit is actually ∞ almost surely. This, however, follows from the assumption. We can split the event $A := \{\lim_{n\to\infty} \tau_n = c < \infty\}$ into the two cases

$$A_1 := \{\lim_{n\to\infty} \tau_n = \tau < \infty \text{ and for all } n, \tau_n < \tau\},$$

$$A_2 := \{\lim_{n\to\infty} \tau_n = \tau < \infty \text{ and for some } n, \tau_n = \tau\}.$$

Note that, by right-continuity and definition of the τ_n, $|\mathbf{X}_{\tau_n} - \mathbf{v}| \le \frac{1}{n}$ whenever $\tau_n < \infty$. Therefore, in the first case, we must have that $\mathbf{X}_{\tau-} = \lim_{n\to\infty} X_{\tau_n} = \mathbf{v}$, which by assumption means that A_1 happens with probability 0. Likewise, A_2 implies that $|\mathbf{X}_\tau - \mathbf{v}| \le \frac{1}{n}|$ for all large n, and so $\mathbf{X}_\tau = \mathbf{v}$, which again has probability 0. Thus the previously given decomposition holds at all times almost surely.

Exercise 2.3

(a) Let x be a càdlàg function on $[0,1]$, and let π^n be a refining sequence of dyadic rational partitions of $[0,1]$ with $\lim_{n\to\infty} \operatorname{mesh}(\pi^n) = 0$. Show that, if the sum

$$\sum_{t_k^n, t_{k+1}^n \in \pi^n} y(t_k^n)(x(t_{k+1}^n) - x(t_k^n))$$

converges to a finite limit for every càglàd function on $[0,1]$, then x is of finite variation.

(b) Let X be a good integrator, and let Π^n be a sequence of partitions tending to identity. Show that

$$\sum_{\tau_k^n, \tau_{k+1}^n \in \Pi^n} Y_{\tau_k^n}(X_{\tau_{k+1}^n} - X_{\tau_k^n}) \overset{\text{ucp}}{\to} (Y \bullet X)$$

for every adapted càglàd process Y.

Solution 2.3

(a) The family of càglàd functions $L([0,1])$ equipped with the uniform norm is a Banach space. For $y \in L([0,1])$, let

$$T_n(y) := \sum_{t_k^n, t_{k+1}^n \in \pi^n} y(t_k^n)(x(t_{k+1}^n) - x(t_k^n)).$$

For each n, pick any $y \in L([0,1])$ such that $y(t_k^n) = \operatorname{sign}\left(x(t_{k+1}^n - x(t_k^n)\right)$ and $||y||_\infty = 1$ (such a y clearly exists). Then we have

$$T_n(y) = \sum_{t_k^n, t_{k+1}^n \in \pi^n} |x(t_{k+1}^n) - x(t_k^n)|.$$

Therefore

$$||T_n|| \geq \sum_{t_k^n, t_{k+1}^n \in \pi^n} |x(t_{k+1}^n) - x(t_k^n)|,$$

for each n, and

$$\sup_n ||T_n|| \geq \text{the total variation of } x.$$

On the other hand, for each $y \in L([0,1])$, $\lim_{n\to\infty} T_n(y)$ exists and therefore $\sup_n |T_n(y)| < \infty$. By the Banach-Steinhaus theorem, we have $\sup_n ||T_n|| < \infty$, and the total variation of x is finite.

(b) Recall that a sequence of partitions tending to identity consists of a sequence of collections of stopping times $\Pi^n := \{0 = \tau_0^n < \ldots < \tau_k^n < \infty\}$ such that $\sup_k |\tau_{k+1}^n - \tau_k^n| \to 0$ and $\sup_l \tau_k^n \to \infty$ as $n \to \infty$. For such Π^n and a given $Z \in \mathbb{L}$, denote

$$Z^{\Pi^n} := \sum_{\tau_k^n, \tau_{k+1}^n \in \Pi^n} Z_{\tau_k^n} \mathbb{1}_{(\tau_k^n, \tau_{k+1}^n]}.$$

Since $\mathbb{S}$ is an ucp-dense subset of $\mathbb{L}$ and $Y \in \mathbb{L}$, we find a sequence (Y^m) of bounded simple processes such that $Y^m \overset{\text{ucp}}{\to} Y$. Then we can rewrite

$$((Y - Y^{\Pi^n}) \bullet X) = ((Y - Y^m) \bullet X) + ((Y^m - (Y^m)^{\Pi^n}) \bullet X) + (((Y^m)^{\Pi^n} - Y^{\Pi^n}) \bullet X).$$

Since $d((Y^m)^{\Pi^n}) \leq d(Y^m, Y)$ for all n, the continuity of the map $Y \mapsto (Y \bullet X)$ (i.e. the fact that X is a good integrator) gives that $((Y^m - Y) \bullet X) \overset{\text{ucp}}{\to} 0$ and $(((Y^m)^{\Pi^n} - Y^{\Pi^n}) \bullet X) \overset{\text{ucp}}{\to} 0$

as $m \to \infty$. Moreover, the observation that $d((Y^m)^{\Pi^n}) \le d(Y^m, Y)$ for all n means that this convergence is uniform in n.

We are then left with the middle term to deal with. As we will see, this is an easier task since we only need to work with simple processes. We know that $Y^m \in \mathbb{S}$ can be written in the form

$$Y_s^m(\omega) = Y_0^m(\omega) + \sum_{i=1}^{M} Y_{\sigma_i(\omega)} \mathbb{1}_{(\sigma_i(\omega), \sigma_{i+1}(\omega)]}(s).$$

(note that both M and the σ_i depend implicitly on m).

Then,

$$(Y^m)_s^{\Pi^n}(\omega) = Y_0^m(\omega) + \sum_{j=1}^{k} Y^m_{\tau_j^n} \mathbb{1}_{(\tau_j^n(\omega), \tau_{j+1}^n(\omega)]}(s)$$

and denote $t_i^n(\omega) := \inf\{\tau_k^n(\omega) : \tau_k^n(\omega) > \sigma_i(\omega)\}$ for each $i = 1, ..., M$. With this choice, note that, on each interval $(\sigma_i(\omega), \sigma_{i+1}(\omega)]$, $Y^m(\omega)$ and $(Y^m)^{\Pi_n}(\omega)$ can only differ on $s \in (\sigma_i(\omega), t_i^n(\omega)]$, since afterwards $(Y^m)^{\Pi_n}$ is “reset” to have the same value as Y^m. Moreover, on $s \in (\sigma_i(\omega), t_i^n(\omega)]$ there is a constant difference between $Y^m(\omega)$ and $Y(Y^m)^{\Pi_n}(\omega)$ of at most $2||Y||_\infty$. Therefore, this inequality follows:

$$\sup_{s \le t} |((Y^m - (Y^m)^{\Pi^n}) \bullet X)_s(\omega)| \le 2||Y||_\infty \sum_{i=1}^{M} |X_{t_i^n(\omega)} - X_{\sigma_i(\omega)}|.$$

Then, right-continuity of X, the bound on Y and the fact that the mesh goes to identity almost surely gives that

$$\sup_{s \le t} |((Y^m - (Y^m)^{\Pi^n}) \bullet X)_s(\omega)| \to 0$$

as $n \to \infty$ almost surely, and thus also in probability.

This yields the result we want. More concretely, we can choose large enough m such that $((Y - Y^m) \bullet X)$ and $(((Y^m)^{\Pi^n} - Y^{\Pi^n}) \bullet X)$ are close to 0 in ucp topology, the latter uniformly over n. Then, we can pick n large enough that the middle term is also close to 0 in ucp topology. This completes the proof.