Mathematical Finance

Exercise sheet 5

Exercise 5.1 We say that S is a special semimartingale if it can be decomposed as

$$S_t = S_0 + M_t + A_t,$$

where M is a local martingale and A is a predictable process of finite variation.

(a) Show that if S is a special semimartingale then the process $Y_t = \sup_{u\in[0,t]} |S_u - S_0|$ is locally integrable.

(b) Give an example of a semimartingale S which is not special.

Solution 5.1

(a) It is clearly enough to do this separately for M and A.

- For M a local martingale: We can, by localisation, assume that M is a martingale. Let $\tau_n := \inf\{t > 0 : |M_t| > n\}$, then we know that M^{τ_n} is again a martingale. Moreover, note that
$$\sup_{s\in[0,t]} |M^{\tau_n}|_s \le n + M_t^{\tau_n}.$$
since the bound of n can only fail at the last time point. But since M^{τ_n} is a martingale, $\sup_{0\le s\le t} |M^{\tau_n}|_s$ is integrable, so that $\sup_{s\in[0,t]} |M_s|$ is locally integrable (since the τ_n clearly go to ∞).
- For A a predictable process of finite variation started at 0: Let $\tau_n := \inf\{t > 0 : |A_t| \ge n\}$. Then τ_n is a predictable time for each n, and we can find an announcing sequence $(\sigma_{n,p})_{p\in\mathbb{N}}$ for τ_n, i.e. a sequence of stopping times such that $\sigma_{n,p} < \tau_n$ a.s. on $\{\tau_n > 0\}$ and $\sigma_{n,p} \nearrow \tau_n$ a.s.. We can find a diagonal subsequence σ_{n,p_n} such that $P(\sigma_{n,p_n} < \tau_n - 1) \le \frac{1}{2^n}$, and let $T_n = \max_{m=1,\dots,n} \sigma_{m,p_m}$.
Clearly, each A^{T_n} is bounded by n as $T_n < \tau_n$ on $\{\tau_n > 0\}$. It is then clear that $\sigma_{n,p_n} \nearrow \infty$ a.s. by construction and since the τ_n do, which shows the result.

(b) We can already find such an example in question 1 of sheet 3. Indeed, this process is a semimartingale (as it is of finite variation) but not locally integrable, by the same proof as used there.

Exercise 5.2 Let $W = W_0 + (W^1, W^2, W^2)$ be a Brownian motion in $\mathbb{R}^3$, i.e. W^1, W^2, W^3 are independent Brownian motions and $W_0 \in \mathbb{R}^3 \setminus \{0\}$ is an $\mathcal{F}_0$-measurable random variable.

(a) Show that $Y_t = |W_t|^{-1}$ is a local martingale as well as a supermartingale, where $|(x,y,z)| = \sqrt{x^2+y^2+z^2}$ is the Euclidean norm. You may assume that $P(\forall t\ W_t \ne 0) = 1$.

(b) Assume that W_0 is a standard normal random variable on $\mathbb{R}^3$. Show by direct calculation that $E[Y_t^2] = \frac{1}{t+1}$ for $t > 0$.

(c) Using the martingale convergence theorem, conclude that Y is not a martingale.

Solution 5.2

(a) By Itô's formula (which we can apply locally, c.f. exercise 2(c) of sheet 2), we obtain the simplified expression

$$dY_t = \sum_{i=1}^{3} -Y_t^3 W_t^i dW_t^i + \frac{1}{2}\sum_{i=1}^{3}\left(3Y_t^5 (W_t^i)^2 - Y_t^3\right) dt.$$

But since $\sum_{i=1}^3 (W_t^i)^2 = Y^{-2}$, the dt term vanishes.

Therefore, Y can be given as an integral against a local martingale (W), and since the integrand is continuous, it is itself a local martingale (c.f. exercise 1(a) of sheet 3). Moreover, Y is non-negative so it is a supermartingale.

(b) Writing $W_0 = (Z^1, Z^2, Z^3)$, we can use independence to find the equality in law:

$$\begin{aligned} |W_t|^2 &= (Z^1 + W_t^1)^2 + (Z^2 + W_t^2)^2 + (Z^3 + W_t^3)^2 \\ &\overset{\mathrm{d}}{=} (t+1)\left((Z^1)^2 + (Z^2)^2 + (Z^3)^2\right). \end{aligned}$$

This is clear since $Z^1 + W_t^1 \sim \mathcal{N}(0, t+1)$, etc.

Now, we note that the sum of the squares of three independent standard normals is a χ_3^2 distribution, or equivalently a Gamma $\left(\frac{3}{2}, \frac{1}{2}\right)$ distribution. Therefore, Y_t^2 is the inverse of a Gamma $\left(\frac{3}{2}, \frac{1}{2(t+1)}\right)$ random variable, so

$$\begin{aligned} E[Y_t^2] &= \int_0^\infty \frac{1}{u} \frac{1}{2^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)} u^{\frac{1}{2}} e^{-\frac{u}{2}}\, du \\ &= \int_0^\infty \frac{1}{u} \frac{1}{2^{\frac{3}{2}}(t+1)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)} u^{\frac{1}{2}} e^{-\frac{u}{2(t+1)}}\, du \\ &= \frac{1}{2(t+1)^{\frac{1}{2}}} \int_0^\infty \frac{1}{2^{\frac{1}{2}}(t+1)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)} u^{-\frac{1}{2}} e^{-\frac{u}{2(t+1)}}\, du \\ &= \frac{1}{t+1}. \end{aligned}$$

(c) From (b) we see that Y is bounded in L^2, and in particular uniformly integrable. Suppose that Y is a martingale. It is then a uniformly integrable martingale, and converges in L^1 to some Y_∞ with $Y_t = E[Y_\infty \mid \mathcal{F}_t]$ for each t. Since Y is bounded in L^2, the convergence holds also in L^2. But then $E[Y_\infty^2] = \lim_{t\to\infty} E[Y_t^2] = 0$, and as Y is non-negative, $Y_\infty = 0$ a.s.. This leads to a contradiction since $E[Y_\infty] = 0 \neq Y_0$.

Exercise 5.3 Let X_k be independent Bernoulli random variables with $P(X_k = +1) = P(X_k = -1) = \frac{1}{2}$, $k \in \mathbb{N}$. Consider an infinite horizon model with a constant bank account normalized to one and a stock $S = (S_k)_{k\in\mathbb{N}}$ whose price is given by $S_0 = 1$, $S_k = S_{k-1} + X_k,\ k \in \mathbb{N}$. Consider the following strategy. Start with zero initial wealth, buy one stock and keep doubling your stock holdings until the stock goes up for the first time, then sell the stocks.

(a) Find the self-financing strategy $\varphi = (\eta, \vartheta)$ and the associated wealth process $V = (V_k(\varphi))_{k\in\mathbb{N}}$ with zero initial wealth for this strategy.

(b) Show that with this strategy, $V_\infty(\varphi) := \lim_{k\to\infty} V_k(\varphi) = 1$ a.s..

(c) Put

$$Y_t := \begin{cases} V_k(\varphi) & \text{for } 1 - \frac{1}{k+1} \leq t < 1 - \frac{1}{k+2}, \\ 1 & \text{for } t \geq 1. \end{cases}$$

Assume the (augmented) natural filtration and show that the process Y is a local martingale, but not a martingale.

Solution 5.3

(a) Define the stopping time

$$\tau = \inf\{k \geq 1 : S_k - S_{k-1} = 1\}.$$

The trading strategy ϑ is given by

$$\vartheta_k = 2^{k-1}\mathbb{1}_{\{\tau > k-1\}}, \; k \in \mathbb{N}.$$

The self-financing strategy $\varphi = (\eta, \vartheta)$ associated to $(V_0, \vartheta) = (0, \vartheta)$ is determined by the value process

$$\begin{aligned} V_k(\varphi) &= \sum_{j=1}^{k} 2^{j-1}\mathbb{1}_{\tau > j-1}(S_j - S_{j-1}) \\ &= \mathbb{1}_{\{\tau > k\}} \sum_{j=1}^{k} 2^{j-1}(-1) + \mathbb{1}_{\tau \leq k} \sum_{j=1}^{\tau} 2^{j-1}\mathbb{1}_{\{\tau > j-1\}}(S_j - S_{j-1}) \\ &= \mathbb{1}_{\{\tau > k\}}(1 - 2^k) + \mathbb{1}_{\{\tau \leq k\}} \left(\sum_{j=1}^{\tau-1} 2^{j-1}(-1) + 2^{r-1}(+1) \right) \\ &= \mathbb{1}_{\{\tau > k\}}(1 - 2^k) + \mathbb{1}_{\{\tau \leq k\}}, k \in \mathbb{N}. \end{aligned}$$

(b) Since $(V_k(\varphi))_{k \in \mathbb{N}}$ is a martingale and $V_0 = 0$, we obtain

$$0 = E[V_k(\varphi)] = (1 - 2^k)P(\tau > k) + P(\tau \leq k).$$

Solving for $P(\tau \leq k) = \frac{2^k - 1}{2^k}$ and letting $k \to \infty$ yields that $P(\tau < \infty) = 1$. Since $P(\tau < \infty) = 1$, we obtain that

$$V_\infty(\varphi) = \lim_{k \to \infty} V_k(\varphi) = 1.$$

(c) Since $E[Y_1] = 1 \neq E[Y_0] = 0$, the process Y is not a martingale. To show that it is a local martingale, choose $\tau_n = \inf\{t \geq 0 : |Y_t| \geq n\} \wedge n$, $n \geq 1$. Let $s < t < 1$ and $A \in \mathcal{F}_s$. We have

$$E[\mathbb{1}_A(Y_t^{\tau_n} - Y_s^{\tau_n})] = E[\mathbb{1}_A(Y_{(\tau^n \wedge t) \vee s} - Y_s)].$$

By the optional stopping theorem, $(Y_t^{\tau_n})_{t \geq 0}$ is a uniformly bounded martingale on $t < 1$. Moreover, Y^{τ_n} is continuous at $t = 1$, and constant on $t \geq 1$. Therefore, $(Y_t^{\tau_n})_{t \geq 0}$ is a martingale, for every $n \geq 1$. This shows that Y is a local martingale.

Exercise 5.4 Consider a financial market on the time interval $[0, T]$ consisting of one numéraire process with constant value $B_t = 1$ and two risky assets with dynamics given by geometric Brownian motion, i.e.

$$\begin{aligned} dS_t^1 &= S_t^1(\mu_1 dt + \sigma_1 dW_t^1), \quad S_0^1 = 1 \\ dS_t^2 &= S_t^2(\mu_2 dt + \sigma_2 dW_t^2), \quad S_0^2 = 1, \end{aligned}$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$ and W^1, W^2 are two Brownian motions with $[W^1, W^2]_t = \rho t$, for any $t \in [0, T]$ and a fixed $\rho \in [-1, 1]$.

(a) Find a condition on the coefficients μ_i, σ_i, ρ that is equivalent to this financial market being arbitrage-free.

(b) Under such conditions, and assuming that the filtration is the one generated by W^1 and W^2, find all the equivalent martingale measures for this financial market.

(c) (optional) Can you repeat this exercise if we assume that the correlation ρ is random and time-dependent, i.e. $[W^1, W^2]_t = \int_0^t \rho_s ds$ for some continuous process ρ_s taking values in $[-1, 1]$?

Solution 5.4

(a) The market is arbitrage-free if and only if either $\rho \notin \{-1, 1\}$ or $\rho \in \{-1, 1\}$ and $\frac{\mu_1}{\sigma_1} = \frac{\rho\mu_2}{\sigma_2}$.

To prove this, suppose that $\rho \notin \{-1, 1\}$. We can easily check that $B = \frac{W^2 - \rho W^1}{\sqrt{1-\rho^2}}$ is a Brownian motion by Lévy's characterisation, i.e.

$$
\begin{aligned}
[B]_t &= \frac{[W^2 - \rho W^1]_t}{1-\rho^2} \\
&= \frac{t - 2\rho^2 t + \rho^2 t}{1-\rho^2} \\
&= t.
\end{aligned}
$$

We can even say more: (W^1, B) is a two-dimensional Brownian motion, again by Lévy's characterisation (by checking that the quadratic covariance vanishes). In particular, W^1 and B are independent.

Now, consider the exponential martingale

$$
Z = \mathcal{E}\left(-\frac{\mu_1}{\sigma_1}W^1 - \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\mu_2}{\sigma_2} - \frac{\rho\mu_1}{\sigma_1}\right)B\right),
$$

which is a martingale, as an easy consequence of Novikov's criterion. Moreover, it is clear from Girsanov's theorem that $\frac{\mu_1}{\sigma_1}t + W_t^1$ and $\frac{\mu_2}{\sigma_2}t + W_t^2$ under the equivalent martingale measure Q with $\frac{dQ}{dP} = Z_T$. Thus, S^1 and S^2 are local martingales under Q, since the integrands are continuous.

Therefore, the market is arbitrage-free. Suppose now that $\rho \in \{-1, 1\}$ and $\frac{\mu_1}{\sigma_1} = \frac{\rho\mu_2}{\sigma_2}$. Then it is easy to check that

$$
Z = \mathcal{E}\left(-\frac{\mu_1}{\sigma_1}W^1\right)
$$

is an exponential martingale and, by Girsanov, we obtain that S^1 and S^2 are again local martingales under Q given by $\frac{dQ}{dP} = Z_T$.

Conversely, suppose $\rho = 1$ and $\frac{\mu_1}{\sigma_1} < \frac{\mu_2}{\sigma_2}$ (other cases, with $\rho = -1$ or the inequality reversed, are similar). Then, we can find an arbitrage strategy with $\vartheta_t^1 = -\frac{1}{\sigma_1 S_t^1}$ and $\vartheta_t^2 = \frac{1}{\sigma_2 S_t^2}$, since then we obtain

$$
\begin{aligned}
G_T(\vartheta) &= \int_0^T \left(-\frac{1}{\sigma_1 S_t^1}S_t^1(\mu_1 dt + \sigma_1 dB_t^1) + \frac{1}{\sigma_2 S_t^2}S_t^2(\mu_2 dt + \sigma_2 dB_t^1)\right) \\
&= \int_0^T \left(\frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}\right) dt \\
&> 0.
\end{aligned}
$$

Thus, there is arbitrage if and only if $\rho \notin \{-1, 1\}$ and $\frac{\mu_1}{\sigma_1} \neq \frac{\mu_2}{\sigma_2}$.

(b) Suppose that Q is an equivalent martingale measure. Then there exists an $\mathcal{F}_T$-measurable Radon-Nikodym derivative $Z = \frac{dQ}{dP} \in L^1(\Omega, \mathcal{F}_T, P)$. Moreover, $Z > 0$ P-a.s. We can also define the P-martingale $Z_t = E_P[Z \mid \mathcal{F}_t]$, which is positive a.s. (as Z is) and has $Z_0 = E_P[Z] = 1$.

Now, by the Itô martingale representation theorem, we can write

$$Z_t = 1 + \int_0^t (\alpha_s dW_s^1 + \beta_s dW_s^2)$$

for some $(\alpha, \beta) \in L^2_{\text{loc}}(W)$. Since Z is positive a.s., we can write

$$Z = 1 + \int_0^t Z_s \left(\frac{\alpha_s}{Z_s} dW_s^1 + \frac{\beta_s}{Z_s} dW_s^2 \right) = \mathcal{E} \left(\int_0^\cdot \frac{\alpha_s}{Z_s} dW_s^1 + \frac{\beta_s}{Z_s} dW_s^2 \right)_t .$$