Mathematical Finance

Exercise sheet 6

Exercise 6.1 Show that in finite discrete time we have

$$(NA) \implies (NUPBR).$$

Solution 6.1 Let Q be an equivalent martingale measure for P which exists by (NA). By Doob's maximal inequality, for every 1-admissible H, we have

$$Q((H \bullet X)^*_T > K) \leq \frac{E_Q[|(H \bullet X)_T|]}{K} = \frac{2E_Q[(H \bullet X)^-_T]}{K} \leq \frac{2}{K}$$

for $K > 0$ (to see this, note that $H \bullet X$ is a local Q-martingale bounded below, which in finite discrete time implies that it is a Q-martingale, and therefore $E_Q[(H \bullet X)^+_T] - E_Q[(H \bullet X)^-_T] = 0$).

Thus, the family $\{(H \bullet X)_T : H \text{ is 1-admissible}\}$ is bounded in Q and consequently in P, i.e. $(NUPBR)$ holds.

Exercise 6.2 Let X be a Banach space and let Y be a closed point-separating linear subset of the dual space. Show that X, together with the topology making all linear functionals of Y continuous (the $\sigma(X, Y)$ topology) is metrizable if and only if X is finite dimensional.

Solution 6.2 We shall use the following lemma: a linear functional l vanishes at the intersection of zero sets of linear functionals

$$\bigcap_{i=1}^{n} \{l_i = 0\},$$

if and only if there are scalars $\lambda_1, \dots, \lambda_n$ such that $l = \sum_{i=1}^n \lambda_i l_i$.

This can be seen by induction: if $l = 0$ on all of X, then $l = 0$ as a functional. Moreover, if $l = 0$ on $\bigcap_{i=1}^n \{l_i = 0\}$, then either $l_n = 0$ on $\bigcap_{i=1}^{n-1} \{l_i = 0\}$ or not. If it is, then l_n is a linear combination of the l_i ($i \leq n-1$) by induction, and so is l, which vanishes on $\bigcap_{i=1}^n \{l_i = 0\} = \bigcap_{i=1}^{n-1} \{l_i = 0\}$. Otherwise, we can find $x_0 \in \bigcap_{i=1}^{n-1} \{l_i = 0\}$ such that $l_n(x_0) \neq 0$. Consider then the new linear functional defined by

$$\tilde{l}(x) = l(x) - \frac{l(x_0)}{l_n(x_0)} l_n(x) = l\left(x - \frac{l_n(x)}{l_n(x_0)} x_0\right).$$

Then this functional vanishes on $\bigcap_{i=1}^{n-1} \{l_i = 0\}$. This is clear since if $x \in \bigcap_{i=1}^{n-1} \{l_i = 0\}$, then $x - \frac{l_n(x)}{l_n(x_0)} x_0 \in \bigcap_{i=1}^n \{l_i = 0\}$ by the construction, and therefore $\tilde{l}(x) = l\left(x - \frac{l_n(x)}{l_n(x_0)} x_0\right) = 0$. Therefore, by induction $\tilde{l}$ is a linear combination of $l_1, ..., l_{n-1}$, and by construction of $\tilde{l}$ it is clear that l is a linear combination of $l_1, ..., l_n$, concluding the proof of this lemma.

Assume now that X is metrizable with respect to $\sigma(X, Y)$, with a metric d_w. Then, for any $x \in X$ it is clear that

$$\left\{ B^{d_w}\left(x; \frac{1}{n}\right) := \left\{ y \in X : d_w(x, y) < \frac{1}{n} \right\}, n \in \mathbb{N} \right\}$$

is a weak neighbourhood basis for x, by properties of metric spaces. Moreover, the set

$$\{l^{-1}(I) : I \subseteq \mathbb{R} \text{ open}, l \in Y\}$$

is a subbasis for the topology. Therefore, setting $x = 0$, we can find some $l_{n,i}, I_{n,i}$ (where $n \in \mathbb{N}, i \in \{1, 2, ..., k_n\}$) such that

$$B^{d_w}\left(0; \frac{1}{n}\right) \supseteq \bigcap_{i=1}^{k_n} l_{n,i}^{-1}(I_{n,i}).$$

Now, take some $l \in Y$, and take $I = (-1, 1)$ an open interval in $\mathbb{R}$. Then $0 \in l^{-1}(I)$. Note that $l^{-1}(I)$ is weakly open, and therefore

$$l^{-1}(I) \supseteq B^{d_w}\left(0; \frac{1}{n}\right) \supseteq \bigcap_{i=1}^{k_n} l_{n,i}^{-1}(I_{n,i})$$

for some n. In particular,

$$l^{-1}(I) \supseteq \bigcap_{i=1}^{k_n} l_{n,i}^{-1}(\{0\}).$$

Take $y \in \bigcap_{i=1}^{k_n} l_{n,i}^{-1}(\{0\})$. Then for any $\lambda \in \mathbb{R}$, $\lambda y \in \bigcap_{i=1}^{k_n} l_{n,i}^{-1}(\{0\})$ and therefore $l(\lambda y) \in I$. But this cannot hold unless $l(y) = 0$, since otherwise we could take λ to be arbitrarily large and obtain a contradiction.

Therefore, l vanishes on $\bigcap_{i=1}^{k_n} l_{n,i}^{-1}(\{0\})$, and therefore belongs to the linear span of $\{l_{n,1}, ..., l_{n,k_n}\}$. But l is arbitrary, and therefore all of Y is spanned by

$$\{l_{n,i} : n \in \mathbb{N}, i \in \{1, 2, ..., k_n\}\}.$$

This is a countable algebraic spanning set, and since Y is a Banach space, this is only possible if Y is finite-dimensional, concluding the proof.

Exercise 6.3 Let $C \subseteq X$ be a convex subset of X, a Banach space. Show that C is closed in X if and only if it is closed with respect to the weak topology $\sigma(X, X^*)$.

Solution 6.3 If $U \neq \emptyset$ be a weakly open set with $x \in U$, it contains some neighbourhood of the form

$$\bigcap_{k=1}^{n} \{y \in X : |x_k^*(y) - x_k^*(x)| < \epsilon\}$$

for some $x_k^* \in X^*$ and $\epsilon > 0$. These subsets are strongly open, and so U is strongly open. Thus, if C is weakly closed, $X \setminus C$ is weakly open and we conclude that C is strongly closed.

Suppose now that C is strongly closed. By the Hahn-Banach separation theorem, we can separate C (a closed convex set) from any $\{x\} \in X \setminus C$ (a convex compact set), by some linear functional x_x^* so that $x_x^*(x) > \sup_{y \in C} x_x^*(y)$. Then, we can easily check that

$$C = \bigcap_{x \in X \setminus C} \{z \in X : x_x^*(z) \leq \sup_{y \in C} x_x^*(y)\}.$$

Since each of these half-spaces is weakly closed, we conclude that C itself is weakly closed.

Exercise 6.4 (Python) Let B be a standard Brownian motion motion and consider a market consisting of three assets $S^0 \equiv 1$, $S_t^1 = \exp(B_t)$ and $S_t^2 = \exp\left(\frac{1}{2}B_t\right)$, $t \in [0, T]$, for some $0 < T < \infty$. Verify numerically that the market admits scalable arbitrage.