Mathematical Finance

Exercise sheet 7

Exercise 7.1 Consider the Bachelier model, taking $[0,1]$ as the time interval and where the price of the risky asset is given by

$$S_t = \int_0^t \sigma dB_s.$$

Consider $\mathcal{X}_1 := \{\vartheta \bullet S, \vartheta \in \Theta^1_{adm}\}$, the set of wealth processes produced by admissible strategies.

(a) Show that $\mathcal{X}_1$ has the concatenation property: for any bounded, predictable $H, G \geq 0$ with $HG = 0$ and for any $X, Y \in \mathcal{X}_1$, if

$$Z = (H \bullet X) + (G \bullet Y) \geq -1$$

then $Z \in \mathcal{X}_1$.

(b) Show that $\mathcal{X}_1$ is closed in the Emery topology.

Solution 7.1

(a) Take any $X = \vartheta_1 \bullet S$, $Y = \vartheta_2 \bullet S$ and bounded, predictable $H, G \geq 0$ with $HG = 0$. Consider then

$$Z = (H \bullet X) + (G \bullet Y) = \sigma(H\vartheta_1 + G\vartheta_2) \bullet B = (H\vartheta_1 + G\vartheta_2) \bullet S,$$

and suppose $Z \geq -1$. But then, $Z \in \mathcal{X}_1$ since $H\vartheta_1 + G\vartheta_2$ is predictable and thus 1-admissible (as $Z \geq -1$).

(b) Assume w.l.o.g. that $\sigma = 1$. Consider the distance $d'(X,Y) = E[1 \wedge [X-Y]_T]$. We can check that if X^n converges to Z in Emery topology, it also converges to Z in d'. Indeed, it is clear that all the X^n are continuous (as integrals against Brownian motion), and so is Z, by the Emery convergence. Moreover, we have that for $a > 0$,

$$P([X^n - Z]_1 > a) \leq P\left((X^n - Z)_1^2 > \frac{a}{2}\right) + P\left(|((X^n - Z) \bullet (X^n - Z))_1| > \frac{a}{4}\right).$$

Since Emery convergence implies ucp convergence, the first probability is small for large enough n. For the second term, we can let $\tau := \inf\{s \in [0,1] : |X^n - Z|_s > 1\}$. Then $(X^n - Z)^\tau$ is bounded by 1, and

$$P\left(|((X^n - Z) \bullet (X^n - Z))_1| > \frac{a}{4}\right) \leq P(\tau \neq \infty) + P\left(|((X^n - Z)^\tau \bullet (X^n - Z))_1| > \frac{a}{4}\right).$$

The first term here is small (again by ucp convergence), and the second is small by Emery convergence.

Now, we obtain that X_n is a Cauchy sequence with respect to d', so that

$$\int_0^1 (\vartheta^n_s - \vartheta^m_s)^2 ds \to 0$$

in probability as $n, m \to 0$, where $X^n = \vartheta^n \bullet S$. We can therefore find some subsequence (we still denote this subsequence by X^n) such that $\int_0^1 (\vartheta^n_s(\omega) - \vartheta^m_s(\omega))^2 ds \to 0$ for almost all

$\omega \in \Omega$. Then for almost all $\omega \in \Omega$, $\vartheta^n(\omega)$ is a Cauchy sequence in $L^2([0,1])$, and contains a limit $\vartheta(\omega)$. It is then clear that

$$\int_0^1 (\vartheta_s^n - \vartheta_s)^2 ds \to 0$$

almost surely. But then, $d'(X^n, X) \to 0$ for $X = \vartheta \bullet S$. Note that this is well defined, since ϑ is predictable (as a limit of the ϑ_n), and $\int_0^1 \vartheta_s^2 ds < \infty$ almost surely by the previous considerations.

Therefore, we easily deduce that $X = Z$. By Emery convergence (implying ucp), it is clear that $Z = X \geq -1$, and therefore $Z = X \in \mathcal{X}_1$.

Exercise 7.2 Suppose we define a model with time interval $[0,1]$, one riskless asset of constant price 1 and one risky asset which is a compound Poisson process with standard normal jumps.

Specifically, for some Poisson process $(N_t)_{t\in[0,1]}$ of rate 1 and $(Z_k)_{k\in\mathbb{N}}$ a sequence of i.i.d. standard normal variables (also independent from N), we have that

$$S_t = \sum_{k=1}^{N_t} Z_k.$$

We take the natural filtration of S.

Show that the only admissible strategy is 0.

Solution 7.2 Let ϑ be an admissible strategy, so that it is predictable and $\vartheta \bullet S \geq -M$ for some $M > 0$ a.s.. Suppose that ϑ is not 0, in the sense that $\vartheta \neq 0$ $dt \times dP$-a.s..

Let $\tau_1, \tau_2, ...$ be the stopping times at which the Poisson process jumps, and take $\tau_0 = 0$. Noting that N has predictable compensator t, we have that

$$E[(|\vartheta| \bullet N)] = E[(|\vartheta| \cdot t)] > 0,$$

by assumption. This means that $E[\mathbb{1}_{\tau_k \leq 1} |\vartheta_{\tau_k}|] > 0$ for some k. In particular we may assume that $P(\tau_k \leq 1, \vartheta_{\tau_k} > \epsilon) > 0$ for some $\epsilon > 0$ (the case $P(\tau_k \leq 1, \vartheta_{\tau_k} < -\epsilon) > 0$ is similar).

Now, note that

$$\begin{aligned} E\left[\mathbb{1}_{\tau_k \leq 1} \mathbb{1}_{\vartheta_{\tau_k} > \epsilon} \mathbb{1}_{Z_k < -\frac{1}{\epsilon}(M + (\vartheta \bullet S)_{\tau_{k-1}})}\right] &= E\left[E\left[\mathbb{1}_{\tau_k \leq 1} \mathbb{1}_{\vartheta_{\tau_k} > \epsilon} \mathbb{1}_{Z_k < -\frac{1}{\epsilon}(M + (\vartheta \bullet S)_{\tau_{k-1}})} \mid \mathcal{F}_{\tau_k -}\right]\right] \\ &= E\left[\mathbb{1}_{\tau_k \leq 1} \mathbb{1}_{\vartheta_{\tau_k} > \epsilon} \Phi\left(-\frac{1}{\epsilon}(M + (\vartheta \bullet S)_{\tau_{k-1}})\right)\right], \end{aligned}$$

where Φ is the cdf of the standard normal distribution. In the above we used that, since ϑ is predictable, ϑ_{τ_k} is $\mathcal{F}_{\tau_k -}$-measurable.

Now, since $P(\mathbb{1}_{\tau_k \leq 1} \mathbb{1}_{\vartheta_{\tau_k} > \epsilon} > 0) > 0$ and $\Phi > 0$, we obtain that the above expectation is positive. We have therefore concluded that

$$P(\tau_k \leq 1, \vartheta_{\tau_k} > \epsilon, Z_k < -\frac{1}{\epsilon}(M + (\vartheta \bullet S)_{\tau_{k-1}})) > 0.$$

But it is clear that this event implies that $(\vartheta \bullet S)_{\tau_k} < -M$. This contradicts the assumption of admissibility.

Exercise 7.3 Consider a general model, with $[0,1]$ as the time interval, a riskless asset of constant price 1, and some d-dimensional semimartingale S representing the prices of the risky assets.

Define

$$G = \{(\vartheta \bullet S)_T, \vartheta \in \Theta_{adm}\} \subseteq L^0$$

and

$$C = (G - L^0_{\geq 0}) \cap L^\infty \subseteq L^\infty.$$

(a) Show that the following notions of no arbitrage are equivalent:

$$G \cap L^0_{\geq 0} = \{0\}$$

and

$$C \cap L^\infty_{\geq 0} = \{0\}.$$

(b) Prove that C is weak-$*$-closed if and only for any bounded sequence (f_n) in C converging almost surely to f_0, it holds that $f_0 \in C$.

Solution 7.3

(a) $\Rightarrow$: Suppose $x \in C \cap L^\infty_{\geq 0}$. By definition of C, there exists some $y \in G$ with $y \geq x \geq 0$. But then, by assumption, $y = 0$ so that $x = 0$.

$\Leftarrow$: Suppose $y \in G \cap L^0_{\geq 0}$. Then let $x = y - (y-1)^+$. Note that $x \in C$ since $(y-1)^+ \in L^0_{\geq 0}$, and $x \in L^\infty$ as $x \leq 1$. If $y \neq 0$ then $x \neq 0$, which contradicts the assumption.

(b) $\Rightarrow$: Suppose x_n are in C and $x_n \to x$ almost surely. Since the x_n are bounded by some $M > 0$, for any $Z \in L^1$, we have that each $|Zx_n| \leq M|Z|$ and so, by DCT, $E[Zx_n] \to E[Zx]$. Therefore $x_n \to x$ in weak-$*$-topology and so $x \in C$.

$\Leftarrow$: Suppose x_n are in C and $x_n \to x$ in weak-$*$ topology. Weak-$*$-convergence implies that the sequence is bounded in L^∞, say by M, and of course also in L^0. By Komlos' lemma, we can find a sequence of forward convex combinations y_n in C with $y_n \to y$ almost surely; moreover the y_n are bounded by some $M > 0$ since the x_n are. But then, for any $Z \in L^1$, we have that each $|Zy_n| \leq M|Z|$ and so, by DCT, $E[Zy_n] \to E[Zy]$. This means that $y_n \to y$ in weak-$*$-topology and so $y = x$. Since y is the almost sure limit of the y_n, which belong to C, then $x = y \in C$ by assumption.

Exercise 7.4 Suppose that $B = \{\xi_\alpha, \alpha \in A\}$ is some family of non-negative random variables, such that for some $\epsilon > 0$ and all $\alpha \in A$,

$$P(\xi_\alpha \geq \epsilon) \geq \epsilon.$$

Show that

$$0 \notin \overline{\mathrm{co}(B)}^{L^0},$$

where $\mathrm{co}(\cdot)$ is the convex hull.

Solution 7.4 Note that for each $\alpha \in A$, splitting the cases $\xi_\alpha \geq \epsilon$ and $\xi_\alpha < \epsilon$,

$$E[e^{-\xi_\alpha}] \leq 1 - \epsilon + \epsilon e^{-\epsilon} < 1.$$

By convexity, the inequality extends to

$$E[e^{-Y}] \leq 1 - \epsilon + \epsilon e^{-\epsilon} < 1$$

for any $Y \in \mathrm{co}(B)$.

If Y_n are in $\mathrm{co}(B)$, and $Y_n \to Y$ in probability, we may assume by taking a subsequence that $Y_n \to Y$ almost surely. Therefore, by DCT we obtain

$$E[e^{-Y}] \leq 1 - \epsilon + \epsilon e^{-\epsilon} < 1$$

for any $Y \in \overline{\mathrm{co}(B)}^{L^0}$.

In particular, 0 does not satisfy this inequality and therefore $0 \notin \overline{\mathrm{co}(B)}^{L^0}$.

Exercise 7.5 (Python) Assume Black-Scholes dynamics for S, say $(r, \mu, \sigma) = (0, 0, 1)$, and find the hedging strategy H for the log-contract g whose discounted payoff is given by

$$g(S_T) = \log \frac{S_T}{S_0} + \frac{1}{2}\sigma^2 T.$$

Compare numerically the value of $g(S_T)$ to $(H \bullet S)_T$ at $T = 1$.