Mathematical Finance

Exercise sheet 8

Exercise 8.1 Consider a probability space $(\Omega, \mathcal{F}, P)$, together with a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$. Consider the natural filtration $\mathcal{F}_t^B = \mathcal{F}_t$ generated by B, and suppose that $\mathcal{F}_T = \mathcal{F}$.

(a) Show that any absolutely continuous measure $Q \ll P$ has a Radon-Nikodym derivative of the form

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds\right)$$

for some $\lambda \in L(B)$.

(b) For Q given in the above form, find (with proof) a d-dimensional Brownian motion under Q.

Remark: You may not use Girsanov's theorem for this part!

Solution 8.1

(a) Let Q be an absolutely continuous measure, with Radon-Nikodym derivative $\frac{dQ}{dP}$. Because $\frac{dQ}{dP}$ is non-negative and integrable, we can by the martingale representation theorem find some $\beta \in L(B)$ such that

$$Z_T := \frac{dQ}{dP} = 1 + \int_0^T \beta_s dB_s.$$

(note that $E[Z_T] = 1$).

Moreover, $\beta \bullet B$ is a martingale and so we have the equality

$$Z_t := E[Z_T \mid \mathcal{F}_t] = 1 + \int_0^t \beta_s dB_s.$$

Note that $Z_t \geq 0$, since the same is true of Z_T. Z is also continuous. Moreover, since Z is a martingale (being a supermartingale suffices for this), we obtain that if $Z_t = 0$ for some $t \geq 0$ then $Z_s = 0$ for all $s \in [t, T]$. This implies that $\beta_s = 0$ for all $s \in [t, T]$.

From these considerations we deduce that we can find $\lambda_s \in L(B)$ such that $\beta_s = \lambda_s Z_s$, and we obtain

$$Z_t = 1 + \int_0^t \lambda_s Z_s dB_s = \mathcal{E}\left(\int_0^\cdot \lambda_s dB_s\right)_t.$$

This yields in particular the result we want.

(b) From Girsanov's theorem, we would expect that $\tilde{B}_t = B_t - \int_0^t \lambda_s^{\mathrm{tr}} ds$ is a Brownian motion under Q. We try to show this directly. Consider some $u \in \mathbb{R}^n$ and some $t \in [0, T]$. Now consider the following:

$$\begin{aligned}
&E_Q\left[\exp\left(i\left(u \cdot B_T^t - \int_0^t u \cdot \lambda_s^{\mathrm{tr}} ds\right) + t\frac{||u||^2}{2}\right)\right] \\
&= E_P\left[\exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds + i\left(u \cdot B_T^t - \int_0^t u \cdot \lambda_s^{\mathrm{tr}} ds\right) + t\frac{||u||^2}{2}\right)\right] \\
&= E_P\left[\exp\left(M_T - \frac{1}{2}\langle M\rangle_T\right)\right] \\
&= 1,
\end{aligned}$$

where $M = \int_0^\cdot \lambda_s dB_s + iu \cdot B^{t_k}$ is a local P-martingale. The last equality holds since $\mathcal{E}(M)$ is a martingale (clearly it is a local martingale, and it is a true martingale by comparison with Z, which we know to be one).

This holds for any u, and by inspecting the first line we conclude that we computed the characteristic function of $\tilde{B}_t$ under Q, and in particular

$$\tilde{B}_t \overset{Q}{\sim} \mathcal{N}(0, tI)$$

where I is the identity matrix.

By a similar computation, we can conclude that the increments of $\tilde{B}$ are independent under Q. Since $\tilde{B}$ is continuous (a.s under P and Q), this shows that $\tilde{B}_t$ is a Brownian motion under Q.

Exercise 8.2 Consider a discrete time setting with deterministic time points $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. In this setting, semimartingales are given in the form

$$S = \sum_{k=0}^{n-1} S_k \mathbb{1}_{[t_k, t_{k+1})} + S_n \mathbb{1}_{\{T\}},$$

where each S_k is $\mathcal{F}_{t_k}$-measurable.

Show that in this case, ucp convergence is equivalent to convergence in Emery topology.

Solution 8.2 It is clear that convergence in Emery topology implies ucp convergence, since the Emery metric is stronger. We can see this by the inequality

$$d_E(S, 0) = \sup_{H \in \mathcal{S}_1} E[1 \wedge (H \bullet S)^*_T] \geq E[1 \wedge S^*_T] = d(S, 0),$$

where d_E and d are the metrics for Emery topology and ucp topology (up to equivalence).

For the converse, note that

$$\begin{aligned} d_E(S, 0) &= \sup_{H \in \mathcal{S}_1} E[1 \wedge (H \bullet S)^*_T] \\ &= \sup_{H \in \mathcal{S}_1} E\left[1 \wedge \sup_{k \in \{0, \ldots, n-1\}} \left| \sum_{i=0}^{k} H_i (S_{i+1} - S_i) \right| \right] \\ &\leq \sup_{H \in \mathcal{S}_1} E\left[1 \wedge \sup_{k \in \{0, \ldots, n-1\}} \sum_{i=0}^{k} |H_i| |S_{i+1} - S_i| \right] \\ &\leq 2n E[1 \wedge S^*_T] \\ &= 2n\, d(S, 0). \end{aligned}$$

Here the n is fixed, therefore we also have that d_E is weaker than d (in this setup), which gives equivalence.

Exercise 8.3 Show that

(a) A local martingale is a sigma-martingale.

(b) A sigma-martingale which is also a special semimartingale is a local martingale.

Solution 8.3

(a) It is sufficient to show that the requirement that the martingale M in the sigma-martingale representation $X = H \bullet M$ with a predictable $H > 0$ can be relaxed to M a local martingale. Let $\tau_0 = 0$ and $(\tau_n)_{n\in\mathbb{N}}$ be the localizing sequence for M in $\mathcal{H}^1$. For each n, set $N^n := \mathbb{1}_{(\tau_{n-1},\tau_n]} \bullet M^{\tau^n}$ and choose $\alpha_n > 0$ such that $\sum_n \alpha_n ||N^n||_{\mathcal{H}^1} < \infty$. Then $N := \sum_n \alpha_n N^n$ is an $\mathcal{H}^1$-martingale and, for $J := \mathbb{1}_{\{0\}} + H \sum_n \alpha_n^{-1} \mathbb{1}_{(\tau_{n-1},\tau_n]} > 0$, we have $X = J \bullet N$.

(b) Let $X = M + A$, where M is a local martingale and A is a predictable FV process with $A_0 = 0$. It is sufficient to show that $A = 0$. There exists predictable $H > 0$ such that $H \bullet X$ is a (local) martingale and without loss of generality we may assume that H is bounded. Indeed, if $X = \tilde{H} \bullet \tilde{M}$ is the sigma-martingale decomposition and $H := \tilde{H}^{-1}$, we have

$$H \bullet X = \tilde{H}^{-1} \bullet (\tilde{H} \bullet M) = \tilde{M}$$

and

$$(H \wedge 1) \bullet X = \left(\frac{H \wedge 1}{H}\right) \bullet (H \bullet X)$$

which is a local martingale, since $H \bullet X$ is. Note that

$$H \bullet A = (H \bullet A)_- + \Delta(H \bullet A) = (H \bullet A)_- + (H \bullet \Delta A),$$

so the process $H \bullet A$ is predictable. Consequently, the process $H \bullet A = H \bullet X - H \bullet M$ is a predictable FV local martingale, so $H \bullet A = 0$. Since A is a FV process, we can decompose $[0, T]$ into two random sets P and N such that $P \cup N = [0, T]$ and $P \cap N = \emptyset$, such that dA is a (non-negative) measure on P and $-dA$ is a (non-negative) measure on N. Taking $J = \mathbb{1}_P - \mathbb{1}_N$ we get

$$\begin{aligned} 0 &= J \bullet (H \bullet A) \\ &= H \bullet (J \bullet A) \\ &= \int_0^{\cdot} H(\mathbb{1}_P dA - \mathbb{1}_N dA) \\ &\geq 0 \end{aligned}$$

as $H > 0$, with equality in the last line if and only if $dA = 0$. Since $A_0 = 0$, this shows the result.

Exercise 8.4 In the same setup of question 1, consider the Bachelier model:

$$S_t = S_0 + \mu t + \sigma B_t$$

on $[0, T]$, where B is a d-dimensional Brownian motion, $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d\times d}$ is invertible.

(a) Show that there exists a unique equivalent martingale measure Q such that for all $f \in L^\infty(\mathcal{F}_T)$, $E_Q(f) = \pi(f)$, where π is the superreplication price.

(b) Take $d = 1$ and $f = (S_T - K)^+$, for some $K \in \mathbb{R}$. Compute $\pi(f)$ as well as the unique strategy ϑ such that

$$\pi(f) + (\vartheta \bullet S)_T = f.$$

(c) Have a look at this paper and write a very short summary of some of the main points.

Solution 8.4

(a) From question 1, any equivalent measure Q can be written in the form

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T ||\lambda_s||^2 ds\right),$$

and then we have that

$$\tilde{B}_t = B_t - \int_0^t \lambda_s^{\mathrm{tr}} ds$$

is a Q-Brownian motion. It is then clear that

$$\frac{dQ}{dP} = \exp\left(\int_0^T -(\sigma^{-1}\mu)^{\mathrm{tr}} dB_s - \frac{T}{2}||\sigma^{-1}\mu||^2\right)$$

is an equivalent martingale measure since $\tilde{B}_t = B_t + \sigma^{-1}\mu t$ is a Brownian motion under Q. Conversely, Q is the unique such measure, since under any other equivalent measure we have that $\tilde{B}$ is a Brownian motion with (non-trivial) drift.

Now, since $\tilde{B}$ is a Brownian motion under Q, and $f \in L^\infty(\mathcal{F}_T, Q)$ (since L^∞ is preserved by an equivalent change of measure), we can use the martingale representation theorem to find some θ such that

$$f = E_Q(f) + \int_0^T \theta_s d\tilde{B}_s = E_Q(f) + \int_0^T \vartheta_s dS_s,$$

where $\vartheta_s = \theta_s \sigma^{-1}$. This means that $E_Q(f) \geq \pi(f)$. Conversely, if we have any representation

$$f \leq f_0 + \int_0^T \tilde{\vartheta}_s dS_s = \int_0^T \tilde{\theta}_s d\tilde{B}_s,$$

with $\tilde{\theta}_s = \tilde{\vartheta}_s \sigma$, we obtain that $f_0 \geq E_Q(f)$, as $\tilde{\theta} \bullet \tilde{B}$ is a local Q-martingale. Thus $E_Q(f) = \pi(f)$.

Now, consider some other martingale measure $\hat{Q}$ with associated process $\hat{\lambda}$. Take

$$f_n = ((\hat{\lambda} + (\sigma^{-1}\mu)^{\mathrm{tr}})\sigma^{-1} \bullet S)_{\tau_n},$$

where $\tau_n := \inf\{t \geq 0 : |((\hat{\lambda} + (\sigma^{-1}\mu)^{\mathrm{tr}})\sigma^{-1} \bullet S)_t| \geq n\} \wedge T$. By continuity and choice of τ_n, f_n is bounded. Note that

$$\begin{aligned}
f_n &= \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s + (\sigma^{-1}\mu)^{\mathrm{tr}})\sigma^{-1} dS_s \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s + (\sigma^{-1}\mu)^{\mathrm{tr}}) d\tilde{B}_s \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s + (\sigma^{-1}\mu)^{\mathrm{tr}}) dB_s + \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s + (\sigma^{-1}\mu)^{\mathrm{tr}})\sigma^{-1}\mu ds \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s\sigma + \mu^{\mathrm{tr}}) d\hat{B}_s + \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s + (\sigma^{-1}\mu)^{\mathrm{tr}})(\sigma^{-1}\mu + \hat{\lambda}_s^{\mathrm{tr}}) ds \\
&= \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s\sigma + \mu^{\mathrm{tr}}) d\hat{B}_s + \int_0^T \mathbb{1}_{[0,\tau_n]}(\hat{\lambda}_s + (\sigma^{-1}\mu)^{\mathrm{tr}})(\sigma^{-1}\mu + \hat{\lambda}_s^{\mathrm{tr}}) ds.
\end{aligned}$$

But then the finite variation term is increasing. Up to localisation, we can take the first term to be a $\hat{Q}$-martingale, and so $E_{\hat{Q}}(f_n) \geq 0$, strictly unless the finite variation term is 0. In order to have equality for all f, and in particular for all f_n, we obtain that $\hat{\lambda}_s = -(\sigma^{-1}\mu)^{\mathrm{tr}}$, which gives uniqueness of Q satisfying the desired properties.

(b) Working under Q, we first want to compute

$$\pi(f) = E_Q[(S_0 + \sigma \tilde{B}_T - K)^+].$$

This is the same as

$$\pi(f) = \sigma\sqrt{T} E_Q\left[\left(\frac{\tilde{B}_T}{\sqrt{T}} - \frac{(K - S_0)}{\sigma\sqrt{T}}\right)^+\right].$$

Letting

$$\psi(x) = \int_x^\infty \frac{(y-x)}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy,$$

we obtain that

$$\pi(f) = \sigma\sqrt{T}\psi\left(\frac{K - S_0}{\sigma\sqrt{T}}\right).$$

More generally, we obtain

$$f_t := E_Q[f \mid \mathcal{F}_t] = \sigma\sqrt{T-t}\,\psi\left(\frac{K - S_0 - \sigma\tilde{B}_t}{\sigma\sqrt{T-t}}\right),$$

for $t < T$. We can then use Itô's formula, as well as the fact that $\psi'(x) = \Phi(x) - 1$ for Φ the cdf of the normal distribution, to obtain

$$df_t = \left(\Phi\left(\frac{K - S_0 - \sigma\tilde{B}_t}{\sigma\sqrt{T-t}}\right) - 1\right)\sigma d\tilde{B}_t,$$

and therefore we obtain the representation

$$f = \pi(f) + \int_0^T \vartheta_s dS_s$$

where

$$\vartheta_t = \left(\Phi\left(\frac{K - S_0 - \sigma\tilde{B}_t}{\sigma\sqrt{T-t}}\right) - 1\right).$$

References

[1] Walter Schachermayer; Josef Teichmann. *How close are the option pricing formulas of Bachelier and Black-Merton-Scholes?* Mathematical Finance, 18: 155-170. doi:10.1111/j.1467-9965.2007.00326.x