Mathematical Finance

Exercise sheet 9

Exercise 9.1 Consider a market model on $[0, T]$ where the risk-free asset has constant value 1, and the risky asset is S. Suppose that $(NFLVR)$ holds.

Show that the following conditions are equivalent:

- For any $f \in L^\infty(\mathcal{F})$, f is attainable.
- There exists a unique ESM Q.

Solution 9.1 $\Leftarrow$: Suppose that there exists a unique ESM Q. If $f \in L^\infty(\mathcal{F})$, we know from the lectures that f can be superreplicated at an optimal price $\pi(f)$, in other words there exists an admissible strategy ϑ^* such that

$$\pi(f) + (\vartheta^* \bullet S)_T \geq f,$$

and moreover

$$\pi(f) = \inf\{c \in \mathbb{R} \mid \exists \vartheta \in \Theta_{\text{adm}} : c + (\vartheta \bullet S)_T \geq f\}.$$

From the lecture notes, we also have that $\pi(f) = \sup_{Q \in \mathbb{P}} E_Q[f]$, where $\mathbb{P}$ is the set of all equivalent σ-martingale measures. Since there is a unique separating measure Q, it follows immediately that

$$\pi(f) = E_Q(f).$$

But now, note that

$$\begin{aligned} E_Q[\pi(f) + (\vartheta^* \bullet S)_T] &= E_Q[E_Q(f) + (\vartheta^* \bullet S)_T] \\ &\leq E_Q(f), \end{aligned}$$

since Q is a separating measure for S. Combined with the super-replication inequality $\pi(f) + (\vartheta^* \bullet S)_T \geq f$, this implies that

$$\pi(f) + (\vartheta^* \bullet S)_T = f,$$

i.e. f is attainable.

$\Rightarrow$: By assumption, $(NFLVR)$ holds and there exists a σ-martingale measure Q for S. Let Q' be another ESM for S.

Note that for any $A \in \mathcal{F}$, we have that $f := \mathbb{1}_A \in L^\infty(\mathcal{F})$ and so it is attainable, i.e.

$$\mathbb{1}_A = c_0 + (\vartheta \bullet S)_T$$

for some constant c_0 and admissible strategy ϑ. We assume that $\vartheta \bullet S$ is bounded. Then, since Q, Q' are ESMs, we have that

$$c_0 \geq Q(A), Q'(A).$$

We can also take expectations for $-\mathbb{1}_A = -c_0 + (-\vartheta \bullet S)_T$, so that

$$-c_0 \geq -Q(A), -Q'(A).$$

But then $Q(A) = c_0 = Q'(A)$ for any $A \in \mathcal{F}$, so that $Q = Q'$.

Exercise 9.2 Consider one risky asset satisfying the SABR model:

$$dS_t = \sigma_t S_t dW_t,$$

where σ, the stochastic volatility, itself satisfies the equation

$$d\sigma_t = \alpha \sigma_t dB_t.$$

We take $\alpha, S_0, \sigma_0 > 0$ and W, B to be two independent Brownian motions.

Compute the superreplication price of European put and call options, $(S_T - K)^+$ and $(K - S_T)^+$, for $K \in \mathbb{R}$.

Solution 9.2 We first prove an upper bound on the superreplication price, which we then show is indeed equal to that price. For a call option $(K - S_T)^+$, note that

$$(K - S_T)^+ \leq K = K + (0 \bullet S)_T.$$

Therefore, $\pi((K - S_T)^+) \leq K$. For a put option, note instead that

$$(S_T - K)^+ \leq (S_T)^+ = S_T = S_0 + (1 \bullet S)_T.$$

Therefore, $(S_T - K)^+ \leq S_0$.

Now, we try to show that these are the true superreplication prices. We take a dual approach. One can easily see, using Novikov's criterion and Girsanov's theorem, that for any $M > 0$ we have an equivalent local martingale measure for S given by the density process

$$Z = \mathcal{E}\left(\int_0^{\cdot} M dB\right) = \mathcal{E}\left(MB\right),$$

i.e. with $\frac{dQ}{dP} = Z_T$.

Next, we show that under Q, $\int_0^T \sigma_t^2$ is large with high probability. Indeed, we have that

$$\sigma = \sigma_0 \mathcal{E}\left(\int_0^{\cdot} (M\alpha ds + \alpha d\tilde{B}_s)\right),$$

where $\tilde{B}$ is a Q-Brownian motion, so that

$$\sigma_t = \sigma_0 \exp\left(M\alpha T + \alpha \tilde{B}_T - \frac{\alpha^2 T}{2}\right).$$

Letting $\epsilon > 0$ be small, it is clear (by applying a Doob's maximal inequality to $\tilde{B}$, say) that we can take M large enough such that

$$Q\left(\sup_{t \in \left[\frac{T}{2}, T\right]} \sigma_t > \frac{1}{\epsilon}\right) > 1 - \epsilon,$$

and in particular we obtain that

$$Q\left(\int_0^T \sigma_s^2 ds > \frac{T}{2\epsilon^2}\right) > 1 - \epsilon.$$

Next, note that

$$S = S_0 \mathcal{E}\left(\sigma \bullet W\right)$$

where W is still a Q-Brownian motion. Therefore, $\sigma \bullet W$ is a continuous local martingale started at 0, and by the Dambis-Dubins-Schwarz theorem we can find a new process $\hat{B}$ which is a (possibly truncated) Brownian motion with respect to Q and its natural filtration, and such that

$$\hat{B}_{\int_0^t \sigma_s^2 ds} = \int_0^t \sigma_s dW_s$$

for each $t \geq 0$.

Let $\tau_T = \int_0^T \sigma_s^2 ds$. By our considerations above, we have that $Q\left(\tau_T > \frac{T}{2\epsilon^2}\right) > 1 - \epsilon$.

Note that $\mathcal{E}(\hat{B})_t \to 0$, Q-a.s. as $t \nearrow \infty$, and therefore for any $\delta > 0$ we can find $t_0 > 0$ such that $Q\left(\sup_{t \geq t_0} \mathcal{E}(\hat{B})_t \geq \delta\right) \leq \delta$. Since we know that τ_T is large with probability close to 1, it follows that we can choose ϵ (hence M) large enough that

$$Q\left(\mathcal{E}(\hat{B})_{\tau_T} \geq \frac{\delta}{S_0}\right) \leq \delta.$$

By our construction of $\hat{B}$, this means that

$$Q(S_T \geq \delta) \leq \delta,$$

in other words, S_T is very small with high probability.

Now, for the put option, note that

$$E_Q[(K - S_T)^+] \geq (K - \delta)(1 - \delta) \nearrow K$$

as $\delta \to 0$, and therefore the superreplication price is at least K. This shows that the superreplication price must be exactly K.

For the call option, the direct argument becomes slightly trickier, but we can use the idea of put-call parity to help. Note that

$$(S_T - K)^+ - (S_T - K)^- = S_T - K,$$

which we can rewrite as

$$(S_T - K)^+ = S_T - K + (K - S_T)^+.$$

Taking expectations under Q,

$$E_Q[(S_T - K)^+] = E_Q[S_T - K + (K - S_T)^+] \geq E_Q[S_T] - O(\delta) \nearrow E_Q[S_T]$$

as $\delta \to 0$. Now, crucially, we need the fact that our constructed Q is indeed a martingale measure for any $M > 0$. This can be checked directly, by applying Novikov's criterion under Q to the local martingale

$$S = S_0 \mathcal{E}\left(\sigma \bullet W\right).$$

Thanks to this fact, we obtain that $E_Q[S_T] = S_0$, and therefore the superreplication price for the call option is S_0.

Exercise 9.3 Consider the model

$$dS_t = \sigma(t, S_t) S_t dW_t, \quad S_0 = s_0 > 0,$$

for some $C^{1,2}$ positive function σ, and assume that there exists a $C^{1,2}$ function f such that $f(t, \cdot)$ is the density of S_t for each $t \geq 0$. Show that

$$\sigma(T, K) = \frac{1}{K} \sqrt{\frac{2 \frac{\partial C}{\partial T}(T, K)}{\frac{\partial^2 C}{\partial K^2}(T, K)}},$$

where $C(t, K) = E[(S_t - K)^+]$ for $K > 0$.
Hint: Consider the value process of some payoff $h(S_T)$, for h smooth enough.

Solution 9.3 Let h be bounded and smooth, say $h \in C_c^\infty((0, +\infty))$. Consider the value process v for h:

$$v(t, x) := E[h(S_T) \mid S_t = x].$$

For $t < T$, we have

$$E[h(S_T)] = E[E[h(S_T) \mid S_t = x]] = \int_0^\infty v(t,x) f(t,x) dx.$$

Differentiating both sides with respect to t yields

$$0 = \int_0^\infty \left[\frac{\partial v}{\partial t} f + v \frac{\partial f}{\partial t} \right] dx.$$

On the other hand, we know that the value process v satisfies

$$\frac{\partial v}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x) x^2 \frac{\partial^2 v}{\partial x^2}(t,x) = 0, \quad v(T,x) = h(x),$$

so we get

$$0 = \int_0^\infty \left[-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} f + v \frac{\partial f}{\partial t} \right] dx.$$

Integrating the first term by parts, we obtain

$$0 = \int_0^\infty v \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2}[\sigma^2 x^2 f] + \frac{\partial f}{\partial t} \right) dx.$$

From the arbitrariness of h (and hence of v), we obtain

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2}[\sigma^2 x^2 f].$$

Finally,

$$\begin{aligned}
\frac{\partial C}{\partial T}(T,K) &= \int_K^\infty (x-K) \frac{\partial f}{\partial T}(T,x) dx \\
&= \frac{1}{2} \int_K^\infty (x-K) \frac{\partial^2}{\partial x^2}[\sigma^2 x^2 f](T,x) dx \\
&= -\frac{1}{2} \int_K^\infty \frac{\partial}{\partial x}[\sigma^2 x^2 f](T,x) dx \\
&= \frac{1}{2}\sigma^2(T,K) K^2 f(T,K) \\
&= \frac{1}{2}\sigma^2(T,K) K^2 \frac{\partial^2 C}{\partial K^2}(T,K),
\end{aligned}$$

from which the formula follows.

Exercise 9.4 Let

$$dY_t = \kappa(\theta - Y_t)dt + \beta\sqrt{Y_t} dW_t, \quad Y_0 = y_0 > 0, \tag{1}$$

where W is a Brownian motion, $\kappa, \theta, \beta > 0$ are constants satisfying the Feller condition $2\kappa\theta > \beta^2$. Show that

$$E\left[\frac{1}{T} \int_0^T Y_t dt \right] = \frac{1 - e^{-\kappa T}}{\kappa T} Y_0 + \left(1 - \frac{1 - e^{-\kappa T}}{\kappa T} \right) \theta. \tag{2}$$

Solution 9.4 We have

$$E[Y_t] = E\left[\int_0^t \kappa(\theta - Y_s) ds + \int_0^t \beta\sqrt{Y_t} dW_s \right] = \kappa\theta t - \kappa E\left[\int_0^t Y_s ds \right].$$

By Fubini, we obtain

$$E[Y_t] = \kappa\theta t - \kappa \int_0^t E[Y_s]ds,$$

i.e. $\xi_t := E[Y_t]$ satisfies the ODE

$$\frac{d\xi_t}{dt} = \kappa\theta t - \kappa\xi_t$$

whose solution with initial solution $\xi_0 = E[Y_0] = Y_0$ is

$$E[Y_t] = \xi_t = Y_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

By Tonelli's theorem, we obtain

$$\begin{aligned} E\left[\frac{1}{T}\int_0^T Y_t dt\right] &= \frac{1}{T}\int_0^T E[Y_t]dt \\ &= \frac{1}{T}\int_0^T (Y_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}))dt \\ &= \frac{1 - e^{-\kappa T}}{\kappa T} Y_0 + \left(1 - \frac{1 - e^{-\kappa T}}{\kappa T}\right)\theta. \end{aligned}$$

Exercise 9.5 (Python) Compute the expectation (2) by simulating the paths of (1).