Partialbruchzerlegung.

Sei $R(x) = \frac{P(x)}{Q(x)}$ eine Rationale Funkhon, und

$$Q(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 = \prod_{k=1}^{N}(x-x_k)^{n_k} \prod_{j=1}^{M}\left((x-\alpha_j)^2 + \beta_j^2\right)^{m_j}$$

Dann gilt

$$R(x) = P_1(x) + \sum_{k=1}^{N} R_k(x) + \sum_{j=1}^{M} S_j(x)$$ wobei

$P_1(x)$ ein Polynom.

$$R_k(x) = \frac{a_{k1}}{(x-x_k)} + \frac{a_{k2}}{(x-x_k)^2} + \dots + \frac{a_{kn_k}}{(x-x_k)^{n_k}}$$

$$S_j(x) = \frac{b_{j1}x + d_{j1}}{\left((x-\alpha_j)^2 + \beta_j^2\right)} + \frac{b_{j2}x + d_{j2}}{(\quad)^2} + \dots + \frac{b_{jm_j}x + d_{jm_j}}{\left((x-\alpha_j)^2 + \beta_j^2\right)^{m_j}}$$

Bsp.

$$R(x) = \frac{1-x}{x^2(x^2+1)}$$

$$= \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{bx+d}{x^2+1}$$

Aus multiplizieren:

$$1 - x = (a_1 + b)x^3 + (a_2 + d)x^2 + a_1 x + a_2$$

Koeffizientenvergleich:

$a_1 + b = 0$

$a_2 + d = 0$

$a_1 = -1$

$a_2 = 1$

$$\frac{x}{x^2+1} - \frac{1}{x^2+1}$$

$$\Rightarrow \int R(x)\,\cancel{dx} = \int -\frac{1}{x} + \frac{1}{x^2} + \overbrace{\frac{x-1}{x^2+1}}\,dx.$$

$$= -\ln|x| - \frac{1}{x} + \frac{1}{2}\ln|x^2+1| - \arctan x + C.$$

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Bsp.

$$R(x) = \frac{4x^3 - 3x + 5}{x^2 - 2x}.$$

$$\begin{array}{l} 4x^3 - 3x + 5 \;\big\lfloor\; x^2 - 2x \\ \underline{-\;4x^3 - 8x^2} \qquad 4x + 8 \\ \quad 8x^2 - 3x + 5 \\ \underline{-\;8x^2 - 16x} \\ \quad\; 13x + 5. \end{array}$$

$$R(x) = 4x + 8 + \frac{13x + 5}{x^2 - 2x}.$$

$$= 4x + 8 + \frac{13x + 5}{x(x-2)}$$

$$\frac{13x+5}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$A(x-2) + B(x) = 13x + 5$$

$$(A + B)x - 2A = 13x + 5$$

$$\left.\begin{array}{l} A + B = 13 \\ -2A = 5 \end{array}\right\} \Rightarrow \begin{array}{l} A = -5/2 \\ B = \frac{31}{2}. \end{array}$$

$$\int R(x)\,dx = \int 4x + 8 - \frac{5}{2}\frac{1}{x} + \frac{31}{2}\frac{1}{x-2}\,dx$$

$$2x^2 + 8x - \frac{5}{2}\ln|x| + \frac{31}{2}\ln|x-2|$$

$$+ c.$$

Bsp. $\dfrac{9}{x^3-3x-2} = \dfrac{9}{(x-2)(x+1)^2}.$

± 1, ± 2, kann die Nullstelle sein.

$$\frac{9}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$9 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

Setze $x=2 \Rightarrow 9 = A\cdot 3^2 = 9A$
$\Rightarrow \boxed{A=1}$

Setze $x=-1 \Rightarrow 9 = C(-3) \Rightarrow \boxed{C=-3}$

Setze $x=0 = 9 = 1\cdot 1^2 + B\cdot(-2)(1) + (-3)(-2)$,

$9 = 1 - 2B + 6 = 7 - 2B$
$\Rightarrow B = -1$

$$\int \frac{9}{x^3-3x-2}\,dx = \int \frac{1}{x-2} - \frac{1}{x+1} - \frac{3}{(x+1)^2}\,dx$$

$$= \ln|x-2| - \ln|x+1| + \frac{3}{x-1} + C.$$

$\int \frac{1}{(x+1)^2}\,dx$ $\qquad$ $x+1 = u$, $dx = du$.

$$\int u^{-2}\,du = \frac{u^{-1}}{-1} = -\frac{1}{x+1}$$

Wir wollen die Stammfunktion von "elementare rationale Funktionen" bestimmen.

z.B.: $\dfrac{1}{(x-a)^n}$, $\dfrac{bx+d}{((x-\alpha)^2+\beta^2)^m}$.

Grundform der Rationaler Funktionen

1) Polynom: $P(x) = \sum_{n=0}^{\cancel{k}} a_n x^n$

$$\boxed{\int P(x)\,dx} = \sum_{n=0}^{k} a_n \frac{x^{n+1}}{n+1} + c,$$

2) Inverse Potenzen.

$$\boxed{\int \frac{dx}{(x-a)^n}} = \begin{cases} \ln|x-a| + c & n=1 \\ \frac{1}{(x-a)^{n-1}}\left(\frac{1}{1-n}\right) & n \geq 2 \end{cases}$$

3) $\boxed{\int \frac{bx+d}{((x-\alpha)^2+\beta^2)^m} \cdot dx} = ?$

Substitution $x-\alpha = \beta t$

$dx = \beta\,dt$

$$\int \frac{bx+d}{((x-\alpha)^2+\beta^2)^m}\,dx$$

$$= \int \frac{b(\beta t+\alpha)+d}{(\beta^2(t^2+1))^m}$$

$$= K \int \frac{Ct+D}{(t^2+1)^m}\,dt$$

$$\int \frac{Ct+D}{(t^2+1)^m}\,dt = C\int \frac{t}{(t^2+1)^m}\,dt + D\int \frac{1}{(t^2+1)^m}\,dt$$

3.1) $\displaystyle\int \frac{t}{(t^2+1)^m}\,dt$

mit Subst. $t^2+1 = u$

$2t\,dt = du.$

$$\frac{1}{2}\int \frac{du}{u^m} = \begin{cases} \frac{1}{2}\ln|u| & m=1 \\ \frac{1}{2}\frac{u^{-m+1}}{1-m}. & m\geq 2 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\ln|t^2+1| + c & m=1 \\ \frac{1}{2}\frac{1}{1-m}\cdot\frac{1}{(t^2+1)^{m-1}} & m\geq 2. \end{cases}$$

3.2) $\displaystyle\int \frac{dt}{(t^2+1)^m}.$

For $m=1$ $\displaystyle\int \frac{dt}{(t^2+1)} = \arctan t + c.$

For $m>1$

Setze $\boxed{I_m := \int \frac{dt}{(t^2+1)^m}.}$

Partielle Integration ergibt.

$$I_m = \int \underbrace{1}_{g'}\cdot\underbrace{\frac{1}{(t^2+1)^m}}_{f}\,dt$$

$$= \frac{t}{(t^2+1)^m} + \int \frac{t\,2mt}{(t^2+1)^{m+1}}\,dt$$

$f = (t^2+1)^{-m}$, $f' = -m\,(t^2+1)^{-m-1}\cdot 2t$

$= -2mt/(t^2+1)^{m+1}$

$$I_m = \frac{t}{(t^2+1)^m} + 2m \underbrace{\int \frac{t^2}{(t^2+1)^{m+1}}\, dt}$$

$$\int \frac{t^2+1-1}{(t^2+1)^{m+1}}\, dt = \underbrace{\int \frac{t^2+1}{(t^2+1)^{m+1}}\, dt} - \int \frac{1}{(t^2+1)^{m+1}}\, dt$$

$$I_m = \frac{t}{(t^2+1)^m} + 2m\left[I_m - I_{m+1}\right].$$

Woraus folgt:

$$2m\, I_{m+1} = \frac{t}{(t^2+1)^m} + (2m-1)\, I_m.$$

$$\boxed{I_{m+1} = \frac{1}{2m}\left[\frac{t}{(t^2+1)^m} + (2m-1) I_m\right]}$$

Rekursive haben wir,

$$I_1 = \arctan t + C.$$

$$I_2 = \frac{1}{2}\left[\frac{t}{(t^2+1)} + \arctan t + C\right].$$

$\vdots$

Das Unbestimmte Integral.

$$\int x^\alpha \, dx = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1} + C & \text{falls } \alpha \neq -1 \\ \ln|x| . & \alpha = -1. \end{cases}$$

$$\int \cos x \, dx = \sin x + C.$$

$$\int \sin x \, dx = -\cos x + C.$$

$$\int \cosh x \, dx = \sinh x + C.$$

$$\int e^x \, dx = e^x + C.$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \, dx = \operatorname{arcsinh} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{arccosh} x + C.$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C.$$

$$\left.\begin{aligned} \int \ln x \, dx &= x \ln x - x + C \\ \int x e^x \, dx &= x e^x - x + C \\ \int x \cos x \, dx &= x \sin x + \cos x + C \end{aligned}\right\} \text{Partielle Integr.}$$

$\int (\text{poly})(\text{trig})$ oder

$\int (\text{poly})(e^x)$ oder $\int (\text{trig})(e^x)$ ..

geeignet für Partielle Integration

Rationale Funktion.

$$\int \frac{P(x)}{Q(x)}\,dx \longrightarrow \text{Partialbruchzerlegung.}$$

<u>Bsp.</u> $\displaystyle\int \frac{1}{(a^2+x^2)^{3/2}}\,dx.$

Substituiere $\quad x = a\tan\theta$.

$\tan\theta = \dfrac{x}{a}, \qquad dx = \dfrac{a}{\cos^2\theta}\,d\theta$

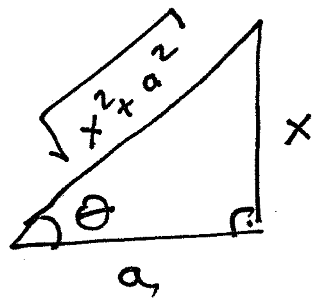
$$\int \frac{1}{\left(a^2(1+\tan^2\theta)\right)^{3/2}}\cdot\frac{a}{\cos^2\theta}\,d\theta.$$

$$\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

$$= \frac{1}{a^2}\int \cos\theta\,d\theta$$

$$= \frac{1}{a^2}\sin\theta + C.$$



$\sin\theta = \dfrac{x}{\sqrt{x^2+a^2}}.$

$$\int \frac{1}{(a^2+x^2)^{3/2}}\,dx = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + C.$$

$$\int \sqrt{1-x^2}\,dx \qquad x = \sin\theta$$

odr $x = \cos\theta$.

$(\sin^2\theta + \cos^2\theta = 1)$.

$(1 - \sin^2\theta = \cos^2\theta)$

Bsp.

$$\int \sqrt{x^2-1}\,dx \qquad x = \cosh t$$

$$dx = \sinh t\,dt$$

$\cosh^2 t - \sinh^2 t = 1$.

$\cosh^2 t - 1 = \sinh^2 t$

$$\int \sqrt{\sinh^2 t} \cdot \sinh t\,dt$$

$$\int \sinh^2 t\,dt = \frac{1}{2}\int (\cosh 2t - 1)\,dt$$

$$= \frac{1}{2}\left[\frac{1}{2}\sinh 2t - t\right] + C.$$

$$\cosh 2t = \cosh^2 t + \sinh^2 t$$

$$= 2\sinh^2 t - 1$$

$$\sinh 2t = 2\sinh t \cdot \cosh t$$

$$\int \sqrt{x^2-1}\,dx = \frac{1}{2}\left[\sinh t \cdot \cosh t - t\right] + C$$

$$= \frac{1}{2}\left[\sqrt{x^2-1} \cdot x - \operatorname{arc\,cosh} x\right] + C.$$

$x = \cosh t$

$\sinh t = \sqrt{x^2-1}$

$$\int \frac{1}{1+\cosh x}\,dx.$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\int \frac{1}{1+\frac{e^x+e^{-x}}{2}}\,dx$$

$$= \int \frac{2}{2+e^x+\frac{1}{e^x}}\,dx$$

$$= \int \frac{2e^x}{2e^x+(e^x)^2+1}\,dx.,$$ $e^x = u$, $e^x dx = du$

$$= \int \frac{2\,du}{u^2+2u+1} = 2\int \frac{du}{(u+1)^2}$$

$$= 2\,\frac{(u+1)^{-1}}{-1} + C$$

$$= -\frac{2}{u+1} + c$$

$$= \frac{-2}{e^x+1} + C.$$