Quick recap:

- Properties of integrals of extended real valued functions:
  - monotonicity (larger function has larger integral)
  - Tchebychev inequality (measure of super level set controlled by integral)
  - L^1 convergence implies convergence in measure and almost everywhere convergence of subsequence
  - "triangle inequality" (put absolute value inside for an upper bound)
  - linearity of summable functions
  - additive over domain of integration
  - (Properly!) Riemann integrable functions are Lebesgue integrable
- Fatou's Lemma: for sequence of non-negative measurable functions, take $\liminf$ out of integral for an upper bound. (Proof: approximations by simple functions from below have a tail satisfying the inequality nearly on whole domain.)
- Monotone Convergence/Beppo Levi's Theorem: For increasing sequence of non-negative measurable functions, limit and integration commute. (Proof: use monotonicity and Fatou's Lemma for the opposite inequality.)
- Dominated Convergence/Lebesgue Theorem: an almost everywhere converging sequence dominated by a summable function converges in L^1. (Proof: apply Fatou's lemma on the dominating function minus the absolute difference of the sequence and the limit, which is non-negative.)
- Differentiation under integral sign on a bounded domain is justified when the integrand has bounded derivative. (Proof: the derivative in absolute value dominates and is summable.)
- Absolute continuity: A summable function has arbitrarily small integral on domains of small enough measure. (Proof: The $\limsup$ of the set of geometrically small measure on which has a positive lower bound has zero measure. Pass to limit using Dominated Convergence Theorem for a contradiction.)
- Vitali's Theorem: On a domain of finite measure, a sequence converges in L^1 if and only if it is uniformly summable and converges in measure. (Main idea: Forward implication uses Tchebychev inequality. In backward implication, a subsequence converges almost everywhere, and by Egoroff's Theorem, it suffices to prove convergence in a set A of small measure. Now, for both implications, observe that one of the quantities $\int_A |f|$, $\int_A |f_k|$ or $\int_A |f - f_k|$ is small if the other two are.)

Exercise 10.1.

(a) For which $s > 0$ is it true that

$$\int_a^b \frac{1}{x^s} dx < \infty \ ,$$

for $(a, b) = (0, 1), (1, \infty), (0, \infty)$?

Guideline:

- Use continuity and non-negativity of x^{-s} to conclude measurability and integrability.
- Find upper and lower bounds by comparing x^{-s} to, say, simple functions like $g(x) = n^s$ for $s \in [\frac{1}{n+1}, \frac{1}{n})$.
- Note that computing the corresponding improper Riemann integrals give the ranges of s but does not prove the results directly.

(b) The Gamma-function is defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \ ,$$

for $s \in \mathbb{R}$, $s > 0$. Show that Γ is everywhere differentiable and calculate its derivative in integral form.

Guideline:

- Formally, differentiate under integral sign to obtain the desired derivative.
- To justify using Dominated Convergence Theorem, find a summable function which dominates $\log(x)e^{-x}x^{s-1}$, uniformly on some interval of s.
- Near infinity, any (mildly) growing exponential controls $\log(x)x^{s-1}$.
- Near the origin, any (small) negative power controls $\log(x)e^{-x}$.

Exercise 10.2.

Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) := \frac{\sin(x)}{x}$. Prove that f is improper Riemann integrable (i.e. the improper Riemann integral exists and is finite), but is not Lebesgue summable.

Hint: You may want to use some results from Analysis 1 and 2. This exercise provides an explicit example for Ex. 8.6.

Guideline:

- To show f is improper Riemann integrable, integrate by parts on any bounded interval to see an integrand with inverse square decay. Convergence is guaranteed by Cauchy's criterion.

- To show f is not Lebesgue summable, obtain a lower bound for $\int |f|$, on the region where $|\sin(x)|$ has a positive lower bound, as a harmonic series.

Exercise 10.3.
Let $\mu(\Omega) < \infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable.

(a) Show that Vitali's Theorem implies Lebesgue's Theorem about dominated convergence.

Guideline:

- Take a sequence converging μ-almost everywhere. Finite total measure implies convergence in measure.

- Uniform μ-summability is guaranteed by the absolute continuity of the integral of the dominating function.

- We need to provide an alternative proof of the absolute continuity without using Dominated Convergence Theorem as in the lecture.

  Theorem: Let $f : \Omega \to \overline{\mathbb{R}}$ μ-summable. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all μ-measurable subsets $A \subset \Omega$ with $\mu(A) < \delta$, it holds $\int_A |f| d\mu < \epsilon$

- Hint: for $f \geq 0$, consider the cut-off function $f_n = \min\{f, n\} \nearrow f$ μ-a.e. and write $|f| \leq |f - f_N| + |f_N|$ for a suitably large N.

(b) Let $\Omega = [0, 1]$. Give an example in which Vitali's Theorem can be applied but there does not exists a dominating function $g \in L^1([0, 1])$.

Hint: Look at $f_k = \chi_{[\frac{k-2^n}{2^n}, \frac{k+1-2^n}{2^n}[}$

Guideline:

- The sequence of functions in the Hint is a typical example that converges in measure but not almost everywhere.

- Multiply a characteristic function of sufficiently small intervals by $1/x$, so that the sequence cannot be dominated by a summable function.

Exercise 10.4.
(Generalized Hölder-inequality) Let $1 \leq p_1, \ldots, p_k \leq \infty$ be given such that $\frac{1}{r} = \sum_{i=1}^k \frac{1}{p_i} \leq 1$. Show that for $f_i \in L^{p_i}(\Omega, \mu)$ it holds $\prod_{i=1}^k f_i \in L^r(\Omega, \mu)$ and

$$\left\| \prod_{i=1}^k f_i \right\|_{L^r} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}}.$$

Guideline:

- Use induction on k.
- Use Hölder's inequality if all $p_i < \infty$.
- If $f \in L^p$ and $g \in L^\infty$, use $\|fg\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^\infty}$.

Exercise 10.5.
Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f_n \xrightarrow{n\to\infty} f$ in $L^p(\Omega, \mu)$ implies

$$\int_\Omega f_n\, g\, d\mu \xrightarrow{n\to\infty} \int_\Omega f\, g\, d\mu$$

for all $g \in L^q(\Omega)$.

Guideline:

- Take the difference and use Hölder's inequality.