For the first three questions it is good to revise the lecture material on Fubini's Theorem and product measures (Chapter IV). The first exercise will test your understanding of the theorem and its proof, whereas the third exercise examines your knowledge of the intricacies of the construction of a product measure. The second exercise is a good way to see an application of Fubini's Theorem. Lastly, the fourth exercise is a revisit of L^p functions which should strengthen your intuition and technical knowledge of the differences between L^p ($p < \infty$) and L^∞ functions.

Exercise 11.1.
Let (X, μ) and (Y, ν) be two measure spaces. Suppose that the functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ are respectively μ-measurable and ν-measurable. Consider the function

$$f \,:\, X \times Y \to \mathbb{R} \qquad \text{with} \qquad f(x,y) \;:=\; g(x)\, h(y)\,.$$

Show that f is $(\mu \times \nu)$-measurable and establish the identity

$$\int_{X\times Y} f\, d(\mu \times \nu) \;=\; \int_X g\, d\mu \cdot \int_Y h\, d\nu\,.$$

Guideline:
Many times when proving results for measurable functions we aim at proving the analogous for simple functions and then conclude by approximation. Thus, it is a good idea to consider simple functions approximating g and h (or their positive and negative parts respectively).

Exercise 11.2.
Let $0 < a < b$. Consider the function x^y on the domain $\{0 \le x \le 1\} \times \{a \le y \le b\}$. By applying Fubini's Theorem to the integral

$$\int_{[0,1]\times[a,b]} x^y dxdy$$

show that

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx = \ln\left[\frac{1+b}{1+a}\right].$$

Guideline:
This is a direct application of Fubini's Theorem. Compute the given integral of x^y in two ways (i.e. by interchanging the integrals).

Exercise 11.3.
Let $X = \mathbb{R}^k$, $Y = \mathbb{R}^l$ be endowed with the measures $\mu = \mathcal{L}^k$, $\nu = \mathcal{L}^l$, respectively the k- and l-dimensional Lebesgue measure.

(a) Show that the product measure $\mu \times \nu$ on $X \times Y = \mathbb{R}^{k+l}$ is given by

$$(\mu \times \nu)(S) = \inf\{(\mu \times \nu)(G) : G \subset \mathbb{R}^{k+l} \text{ is open }, S \subset G\}.$$

(b) Show that $\mu \times \nu$ is just the $(k+l)$-dimensional Lebesgue measure $\mathcal{L}^{k+l}$ on $\mathbb{R}^{k+l}$.

Guideline:

(a) We show that the definition in the lecture for product measures generated by rectangles of the form $A \times B$ is equivalent to the one using open sets. It is enough to show that the product measure is greater or equal then $\inf\{(\mu \times \nu)(G) : G \subset \mathbb{R}^{k+l} \text{ is open }, S \subset G\}$ over open sets. The converse follows by monotonicity. Use the definition in the lectures and the definition of inf to obtain a countable collection of rectangles $\{A_i \times B_i\}_i$ within ε distance of the product measure. How can you then obtain a countable collection of open sets within $C\varepsilon$ distance of the product measure (where $C > 0$ is a constant)?

(b) If there is equality on sets $A \times B$, then one should obtain equality of the measures by the uniqueness of the Caratheodory-Hahn extension.

Exercise 11.4.

(a) Show that any $f \in \bigcap_{p\in\mathbb{N}} L^p(\Omega, \mu)$ with $\sup\limits_{p\in\mathbb{N}} \|f\|_{L^p} < \infty$ lies in $L^\infty(\Omega, \mu)$ as well.

Hint: Tchebychev' inequality.

(b) Show that if $\mu(\Omega) < \infty$, then for any f as in 11.4.a), we have that $\|f\|_{L^\infty} = \lim\limits_{p\to\infty} \|f\|_{L^p}$.

(c) Find an $f \in \bigcap_{p\in\mathbb{N}} L^p(\Omega, \mu)$, where $\mu(\Omega) < +\infty$, with $f \notin L^\infty(\Omega, \mu)$, i.e. show that the result from 11.4.a) does not hold true without the assumption $\sup\limits_{p\in\mathbb{N}} \|f\|_{L^p} < \infty$.

Guideline:

(a) Show that the set where f is not L^∞ has null μ measure.

(c) Consider intervals such as $[0, 1]$. If no x^{-p} is enough to obtain the result, then which function should you pick?