Quick recap:

- The *Hardy–Littlewood maximum function* is the supremum of normalized L^1 norm over balls of a given center. It satisfies a *weak type estimate* analogous to Tchebyshev inequality. (Proof: Around each point of the tail set the average is large over some ball. Cover the tail set with these balls and make them disjoint using Vitali's covering theorem, at a cost of a dimensional multiplicative constant.)

- *Vitali's covering theorem* enables the covering of a family of balls with bounded diameter by the dilation of a disjoint submafily. (Proof: Pick greedily at each dyadic scale. Those not included intersect another and thus are covered by an enlarged version of the other.)

- *Lebesgue differentiation theorem*: a locally integrable function equals its average over infinitesimal ball, almost everywhere (at *Lebesgue points*). (Proof: Locally, approximate with continuous functions and use Tchebyshev/weak type estimate accordingly.)

- ν is *absolutely continuous with respect to* μ is μ-null sets are ν-null. They are *mutually singular* if there is a ν-null Borel set whose complement is μ-null.

- *Radon–Nikodym theorem* states that if ν is absolutely continuous with respect to μ, then ν can be recovered by an μ-integral of the (Radon–Nikodym) *derivative*, or called *density*, which is the their quotient on infinitesimal ball.

- A corollary is the *Lebesgue differentiation theorem for Radon measures.* (Proof: Write the integrand as the Radon–Nikodym derivative.)

- A non-decreasing left-continuous function is *absolutely continuous* if it does not fluctuate on a null set. It implies uniform continuity and is implied by Lipschitz continuity. An equivalent condition is the absolute continuity of the associated Lebesgue–Stieltjes measure with respect to the Lebesgue measure.

Exercise 13.1.
Let $\mathbb{R}^n$ be endowed with the Lebesgue measure $\mathcal{L}^n$ and consider a measurable subset $E \subset \mathbb{R}^n$. Denote by $B_r(x)$ the ball of radius r centered at x. Show that, for $\mathcal{L}^n$-a.e. $x \in E$ it holds

$$\lim_{r\to 0} \frac{\mathcal{L}^n\left(E \cap B_r(x)\right)}{\mathcal{L}^n\left(B_r(x)\right)} = 1.$$

Guidelines: Use Lebesgue differentiation theorem on the characteristic function of E.

Exercise 13.2.
Let μ be a finite Borel measure on $[1, \infty)$ satisfying

(i) $\mu \ll \lambda$ with continuous Radon-Nikodým derivative $d\mu/d\lambda = f$.

(ii) $\mu(B) = \alpha^2 \mu(\alpha B)$ for each $\alpha \geq 1$ and each Borel subset $B \subset [1, \infty)$.

Prove that there exists some nonnegative constant M such that

$$f := \frac{M}{x^3} \qquad \forall\ x \geq 1\,.$$

Guidelines:

- Use the Radon–Nikodym theorem for a useful representation.
- Specialize to the set $[1, x]$.
- Differentiate under the integral sign to get an algebraic equation.
- Fix a point.

Exercise 13.3.
Let $g(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$ for $x \in \mathbb{R}^2 \setminus \{0\}$. Show that $u = f * g$ solves the Laplace-Equation

$$-\Delta u = f$$

for $f \in C_c^\infty(\mathbb{R}^2)$.

Guidelines:

- Outside the support of f, differentiate under the integral sign.
- Inside the support of f, write the integral as a limit of regular integrals.
- Use $\Delta_x f(x-y) = \Delta_y f(x-y)$ and then integrate by parts (Green's identity) to take the limit on the boundary term.

Exercise 13.4.
(a) Is the Hardy-Littlewood maximal function f^* of $f = \chi_{[0,1]}$ summable?

Guidelines: For large x, obtain a lower bound which comes from the normalization factor.

(b) Find the smallest C with $\mathcal{L}_1(\{f^* > \alpha\}) \leq \frac{C}{\alpha}$ for all $\alpha > 0$, with $f = \chi_{[0,1]}$.

Guidelines:

- Consider the cases $x \in [0,1]$, $x \geq 1$, and $x \leq 0$.
- Choose the optimal constant.

Exercise 13.5.
(a) Show: For $f \in L^\infty(\mathbb{R}^n)$ is $f^* \in L^\infty(\mathbb{R}^n)$ and $\|f^*\|_{L^\infty} \leq \|f\|_{L^\infty}$.

Guidelines: Upper bound $|f|$ by the L^∞-norm.

(b) Show that $(f+g)^* \leq f^* + g^*$ for all $0 \leq f, g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Guidelines: Use the triangle inequality.

(c) Find f and g with $(f+g)^*(x) < f^*(x) + g^*(x)$ on a set of positive measure.

Guidelines: Choose two nontrivial locally integrable functions that sum to zero.