Quick recap of lecture materials.

- The *Vitali set* contains (by axiom of choice) exactly one element from each equivalent class in the unit half-open interval, where elements under addition of rational numbers modulo 1 are identified. It is Lebesgue non-measurable.
- The *Cantor Triadic Set* is constructed by successively removing each middle-third interval starting from the closed unit interval. It is uncountable and has Lebesgue measure 0.
- The *Lebesgue–Stieltjes measure* on the real line is a Carathéodory–Hahn extension like the Lebesgue measure, but with the length replaced by the difference of values of an increasing, left-continuous function (when it is the identity function we recover the Lebesgue measure).
- Theorem 1.6.2: A *metric measure* (additive for separated sets) in the Euclidean space is Borel, because any closed set when intersected with an arbitrary set can be shrunk to stay separated with the original complement, and the error is two separated unions of rings.
- Theorem 1.6.4: The Lebesgue–Stieltjes measure is metric hence Borel, and also Borel regular, because disjoint subsets can be covered and well-approximated by disjoint families of intervals.

Exercise 4.1.
Prove that the Lebesgue measure is invariant under translations and rotations, i.e. under all motions of the form
$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(x) = x_0 + Rx,$$
for $x_0 \in \mathbb{R}^n$ and $R \in O(n)$.

Hint: You may use the invariance of the Jordan measure, see Struwe's lecture notes[1]

Guidelines:

- By outer regularity (infimum over measure of open sets), it suffices to check the invariance for open sets.
- By dyadic cube decomposition, any open set is a countable union of intervals.
- By Hint, the Jordan measure of an interval is invariant under translations and rotations.
- Whenever exists, the Jordan measure equals the Lebesgue measure (Theorem 1.3.11).

Exercise 4.2.
Show that every countable subset of $\mathbb{R}$ is a Borel set and has Lebesgue measure zero.

Guidelines:

- Recall the definition of the Borel σ-algebra: generated by open subsets.

[1] see Satz 9.3.2. in Struwe's notes

- Are single points Borel?
- Single points can be covered by arbitrarily small intervals, so they have arbitrarily small non-negative measure, i.e. measure zero.
- For the measure of general countable subsets, use σ-additivity.

Exercise 4.3.
The goal of this exercise is to show that the Cantor triadic set C is uncountable. For that, recall quickly the construction of C: Every $x \in [0,1]$ can be expanded in base 3, i.e. can be written as $x = \sum_{i=1}^{\infty} d_i(x) \frac{1}{3^i}$ for $d_i(x) \in \{0,1,2\}$. The set C is then defined as the set of those $x \in [0,1]$ which do not have any digit 1 in their 3-expansion, i.e.:

$$C := \{x \in [0,1] \mid d_i(x) \in \{0,2\}, \forall i \in \mathbb{N}\}.$$

Now, the Cantor-Lebesgue function F is defined by

$$F : C \to [0,1], \quad F\left(\sum_{i=1}^{\infty} \frac{a_i}{3^i}\right) := \sum_{i=1}^{\infty} \frac{a_i}{2^{i+1}}$$

(a) Show that $F(0) = 0$ and $F(1) = 1$.

Guidelines for (a):

- What are the ternary expansions of 0 and 1, using only the digits $\{0,2\}$?
- Sum the geometric series.

(b) Show that F is well-defined and continuous on C.

Guidelines for (b):

- Every element in C has a unique 3-expansion: The only possible ambiguity is a recurrent 2, which results in a 1.
- Recall the n-th step of construction where the first n digits cannot contain the digit 1.
- If two numbers are in the same interval of C_n, then they differ at most by the tail of a geometric series, which is small.
- If two numbers are close enough, then they must lie in the same interval of C_n for any given large n (because the other side does not contain a point of C).

(c) Show that F is surjective.

Guidelines for (c):

- Every point in $[0,1]$ has a unique binary expansion.
- There is a bijective map between the two set of digits, $\{0,1\}$ and $\{0,2\}$.

(d) Conclude that C is uncountable.

Guidelines for (d):

- $[0,1]$ is uncountable.
- Use parts (b) and (c).

Exercise 4.4.
Let $A \subset \mathbb{R}^n$ and $0 \leq s < t \leq \infty$. Show that:

(a) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.

(b) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.

Observe that this implies that there exists a uniquely determined value $d \in \mathbb{R}$ such that $\mathcal{H}^s(A) = 0$ for $s > d$ and $\mathcal{H}^s(A) = +\infty$ for $s < d$. The value d is called the *Hausdorff dimension* of A.

Guidelines:

- The two statements are contrapositive of each other, hence equivalent.
- Given a covering of A by balls, one can bound the sum of radii raised to power s, independently of the maximum radius.
- When the dimension parameter is increased, reduce the maximum radius and look for a small factor.

Exercise 4.5.

(a) Let $A \subset \mathbb{R}$ be a subset with Lebesgue measure $\mathcal{L}^1(A) > 0$. Show that there exists a subset $B \subset A$ which is **not** $\mathcal{L}^1$-measurable.

Guidelines for (a):

- By translation invariance, assume A has positive measure in the unit interval.
- Consider the measure of the intersection with the rational translations of the Vitali set.
- The measurability of all these sets is inconsistent with the σ-additivity of the Lebesgue measure.

(b) Find an example of a countable, pairwise disjoint collection $\{E_k\}_k$ of subsets in $\mathbb{R}$, such that:

$$\mathcal{L}^1\Big(\bigcup_{k=1}^{\infty} E_k\Big) < \sum_{k=1}^{\infty} \mathcal{L}^1(E_k).$$

Guidelines for (b):

- The strict inequality means non-measurability.
- The Vitali set has positive measure, in order to be Lebesgue non-measurable.
- The Lebesgue measure is translation invariant.