In order to attack this problem sheet, it can be useful to having went through Problem Sheet 4 and the lecture notes on Hausdorff measure and non-integer Haussdorff dimensions.

For the students interested in Probability Theory, Taylor has proven in 1953 (*The Hausdorff α-dimensional measure of Brownian paths in n-space*) that the 1-dimensional Brownian Motion has Hausdorff dimension 3/2, and the d-dimensional Brownian Motion has dimension 2 (for $d \geq 2$).

Warmup Exercise.
What is the Hausdorff measure of a n-unit ball?

Solution. Read Example 1.7.6, Remark 1.7.7. It follows immediately that $\mathcal{H}^n(B^n) = 1$, where $B^n \subset \mathbb{R}^n$ is the n-unit ball. When $n = 1$, we have

$$\mathcal{H}^1(B^1) = \mathcal{H}^1([-1,1]) = 1 = \frac{1}{2}(1-(-1)).$$

Compare the last equality with the result in Exercise 2.

Extra Exercise.
Construct a set X that has Hausdorff dimension 1, but $\mathcal{H}^1(X) = 0$.

Solution. Let $X_n \subset \mathbb{R}$ sets with Hausdorff dimension $1 - 1/n$, $n \geq 1$. Define

$$X = \bigcup_{n\geq 1} X_n.$$

Let $s \geq 1$. Since X_n have Haussdorff dimension $1 - 1/n < s$ for all $n \geq 1$, then $\mathcal{H}^s(X_n) = 0$. We have:

$$\mathcal{H}^s(X) \leq \sum_n \mathcal{H}^s(X_n) = 0.$$

Moreover, if $s < 1$, there exists n_0 such that $1 - 1/n_0 > s$. Hence, by monotonicity we obtain

$$\mathcal{H}^s(X) \geq \mathcal{H}^s(X_{n_0}) = \infty.$$

Source:
`https://math.stackexchange.com/questions/1583133/sets-with-hausdorff-measure-0`

Exercise 5.1.
Let E be the collection of all numbers in $[0,1]$ whose decimal expansion with respect to the basis 10 has no sevens appearing.

Recall that some decimals have two possible expansions. We are taking the convention that no expansion should be identically zero from some digit onward; for example $\frac{27}{100}$ should be written as $0,269999....$ and not as $0,27$.

Prove that E is a Lebesgue-measurable set and determine its Lebesgue measure.

Guidelines

- Maybe it is easier to find the measure of $[0,1] \setminus E$.
- Can you write $[0,1] \setminus E$ as a union of intervals (using the convention above)?
- Related question: How can you make the representation of the Cantor set C (see notes) unique? If $x \in C$ has two representations, you can always assign the representation that has less zeros.

Exercise 5.2.
Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuous injective curve. We define the arc length of γ as:

$$L(\gamma) = \sup \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all $N \in \mathbb{N}$ and all $a \leq t_0 \leq \ldots \leq t_N \leq b$. Show:

$$\mathcal{H}^1(\mathrm{Im}(\gamma)) = \frac{1}{2} L(\gamma)$$

Guidelines This exercise needs a bit more care. Note that unlike the Lebesgue measure, the Hausdorff measure is induced by the sum of powers of radii of covering balls. Thus, the reason why we get $\mathcal{H}^1(\mathrm{Im}(\gamma)) = \frac{1}{2}L(\gamma)$ instead of the full $L(\gamma)$.

- As a useful technique, it is easier to prove an upper bound of $\mathcal{H}^1(\mathrm{Im}(\gamma))$. Thus, show that $\mathcal{H}^1(\mathrm{Im}(\gamma)) \leq \frac{1}{2}L(\gamma)$.
- Next, we need to find a lower bound of $\mathcal{H}^1(\mathrm{Im}(\gamma))$ - we aim to show that $\mathcal{H}^1(\mathrm{Im}(\gamma)) \geq \frac{1}{2}L(\gamma)$. It will be helpful to use and prove the following claim:
  Given $\phi : [a,b] \to \mathbb{R}^n$ continuous curve, show that

  $$2 \cdot \mathcal{H}^1(\mathrm{Im}\ \phi) \geq \mathrm{diam}(\mathrm{Im}\ \phi),$$

  where diam is the canonical definition of the diameter of a set wrt a metric.

Exercise 5.3.

(a) Let I be the interval $[0,1]$ in $\mathbb{R}$. Prove that $\mathcal{H}^1(I) = \frac{1}{2}$.

(b) Perform inductively the following construction: start with a closed equilateral triangular region E_0 of side 1. Let E_1 be made of the three closed equilateral triangular regions T_1^l ($l = 1,2,3$) of side $\frac{1}{3}$ which are inside E_0 and are located in the corners of E_0. In other words we are removing an open hexagon of side $\frac{1}{3}$ from the middle of E_0.
At each step of the construction, we start from a set E_{j-1} which is made of the 3^{j-1} closed triangular regions T_{j-1}^l (for $l = 1, ..., 3^{j-1}$). On each T_{j-1}^l we perform the analogous construction: we take the three closed equilateral triangular regions which are located at the

corners of T^l_{j-1} and have sides of length $\frac{1}{3}$ of the sides of T^l_{j-1}; performing this in each T^l_{j-1} we get 3^j triangles T^l_j, whose union we denote by E_j. Set $E = \cap_{j=1}^{\infty} E_j$.
Show that the Hausdorff dimension of E is 1.

Hint: For each $\delta > 0$ in a sequence converging to 0, find a suitable covering to show that $\mathcal{H}^1(E)$ is bounded from above by a positive number. To get a positive lower bound, observe the projection of E onto one of the sides of E_0. You may want to apply the result given in Exercise 5.4.

Guidelines

- Point a) follows from Exercise 2.
- The delicate part is finding a positive lower bound. You may want to understand the projection map of E onto one of the sides E_0. What is the Hausdorff measure of this projection? Is the projection Lipschitz?

Exercise 5.4.
Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz with constant L. Let $A \subset \mathbb{R}^n$ and $0 \leq s < +\infty$. Show that:

$$\mathcal{H}^s(f(A)) \leq L^s \mathcal{H}^s(A)$$

Exercise 5.5.
Let C denote the Cantor set as defined in the lecture. Show that it holds:

$$\dim_{\mathcal{H}}(C) = \frac{\ln(2)}{\ln(3)} =: s,$$

and that $2^{-s-1} \leq \mathcal{H}^s(C) \leq 2^{-s}$.

Guidelines In order to be comfortable with such exercises, I also recommend showing carefully that $\dim_{\mathcal{H}}(K) = \frac{\ln(4)}{\ln(3)}$, where K is the Koch curve (see lecture notes). Steps:

1. Show $\mathcal{H}^1(K) = \infty$.
2. Show $\mathcal{H}^2(K) = 0$.

This should give you a good understanding of what happens to the Hausdorff dimension. In particular, if we handwave we see that

$$\mathcal{H}^s(K) \approx 4^n \cdot \frac{1}{3^{ns}}.$$

Thus, for $s = \log_3 4 = \frac{\ln(4)}{\ln(3)}$, we obtain that $\mathcal{H}^s(K) \in (0, \infty)$. In particular, that

$$\dim_{\mathcal{H}}(K) = \frac{\ln(4)}{\ln(3)}.$$

Prove the above carefully!

One can repeat a similar line of thought for the Cantor set. It will be more delicate to show:

$$2^{-s-1} \leq \mathcal{H}^s(C) \leq 2^{-s}.$$

In order to obtain the lower bound it is crucial to observe that C is compact and hence you can always assume wlog that C is covered by finitely many open balls. Thus, one can find for each finite cover, a positive lower bound for the radii (carefully chosen!).