As most of this exercise sheet deals with a new and central concept of the second part of the course, namely *measurable functions*, I highly recommend that you read through the notes more than once to get a good feeling for it.

Many exercises use those from past sheets, so it will definitely be useful to go through them if something looks familiar.

Exercise 6.1.
Prove the following claims.

(a) $\mathcal{L}^n$, Λ_F are Radon measures on $\mathbb{R}^n$ and $\mathbb{R}$, respectively.

Guidelines:

- Recall that Borel-regularity was proven in lecture.
- Compact sets are bounded, so their measure can be approximated from above by certain nice sets.

(b) $\mathcal{H}^s$ is not a Radon measure for $s < n$, but it is a Radon measure for $s \geq n$.

Guidelines:

- Recall that $\mathcal{H}^n$ and $\mathcal{L}^n$ are related by a constant.
- The most important property when dealing with multiple Hausdorff measures is the one from Ex. 4.4.

(c) If μ is a Radon measure, $A \subset \mathbb{R}^n$ μ-measurable, then $\mu \llcorner A$ with

$$(\mu \llcorner A)(B) := \mu(A \cap B), B \subset \mathbb{R}^n$$

is a Radon measure as well.

Guidelines:

- Properties of $\mu \llcorner A$ were explored in an earlier exercise. From there it follows easily that the only interesting things is the Borel-regularity of sets of finite measure.
- Let $B \subset \mathbb{R}^n$ be arbitrary. The only thing you have going is that A is μ-measurable, so it is natural use Borel-regularity of μ to approximate $A \cap B$ and $B \backslash A$ with some Borel sets C and D. What Borel set is then a natural choice for approximating $(\mu \llcorner A)(B)$?

Exercise 6.2.
The goal of this exercise is to prove Lemma 1.8.3 in the lecture notes:

Lemma. *Let μ be a Radon measure on $\mathbb{R}^n$. Then for every Borel set $B \subset \mathbb{R}^n$, it holds:*

$$\forall \varepsilon > 0, \exists G \text{ open}, B \subset G : \mu\big(G \setminus B\big) < \varepsilon$$

(a) Show that any closed subset F in $\mathbb{R}^n$ can be expressed as the countable intersection of open subsets containing F.

Guidelines:

- Think about how you would express a singleton as a countable intersection of open sets?
- A very similar idea works for an arbitrary closed set.

(b) Using the fact that $\mathbb{R}^n$ can be exhausted by a countable family of compact subsets, deduce that we can assume that μ is a finite Radon measure by restricting μ to certain bounded subsets.

Guidelines:

- Recall Example 1.8.1 in the lecture notes.
- If the Lemma is true for finite Radon measures, then it is specifically true for compact sets of the chosen exhaustion. This means that you can approximate a part of B contained in such a compact set arbitrarily well by an open set in it. Now, how to combine these approximations?

(c) Let us define:

$$\mathcal{A} := \left\{ B \subset \mathbb{R}^n \mid \forall \varepsilon > 0, \exists G \text{ open}, B \subset G : \mu(G \setminus B) < \varepsilon \right\}$$

Observe that all open subsets lie in $\mathcal{A}$. Check that $\mathcal{A}$ is closed under countable unions.

Guidelines:

- The proof of this is very similar to part b) and the proof strategy should be very familiar to you from past exercise sheets and proof seen in lecture.

(d) Check that $\mathcal{A}$ is closed under countable intersections. Conclude that all closed subsets are in $\mathcal{A}$.

Guidelines:

- The same proof as in part c) works... almost! Where is the problem?
- This is why it was crucial to prove part b). The setting involving a set of finite measure and a limiting process should remind you of a certain early result about properties of measure.

(e) Define $\mathcal{A}' := \left\{ B \subset \mathbb{R}^n \mid B \in \mathcal{A} \text{ and } B^c \in \mathcal{A} \right\}$. Conclude from the previous steps that $\mathcal{A}'$ is a σ-algebra containing all Borel sets. Conclude the proof.

Guidelines:

- This should be straightforward from all the previous results of the exercise.

Exercise 6.3.
Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Show that the following are equivalent.

i) $f^{-1}(U)$ is μ-measurable for every open $U \subset \mathbb{R}$
ii) $f^{-1}(B)$ is μ-measurable for every Borel set $B \subset \mathbb{R}$.
iii) $f^{-1}(]-\infty, a[)$ is μ-measurable for every $a \in \mathbb{R}$.

Guidelines:

- Exercise 2.1 should be helpful here.
- Recall a nice way by which the Borel σ-algebra can be generated.

Exercise 6.4.
Let (X, μ, Σ) be a measure space and $f, g : X \to \mathbb{R}$ two measurable functions on X. Show that the sets $\{x \mid f(x) = g(x)\}$ and $\{x \mid f(x) < g(x)\}$ are measurable.

Guidelines:

- When trying to prove that a certain set is measurable, a general proof strategy is to express it as a preimage of a measurable function.
- In view of the previous bullet point, go through the lecture notes once again to see in what ways can measurable function be combined to give new measurable functions.

Exercise 6.5.
A function $g\colon \mathbb{R} \to \mathbb{R}$ is called Borel measurable, if for every open set $U \subset \mathbb{R}$ the set $g^{-1}(U)$ is a Borel set. Let (X, μ, Σ) be a measure space, let $f\colon X \to \mathbb{R}, g\colon \mathbb{R} \to \mathbb{R}$ be functions with f μ-measurable and g Borel-measurable. Show that $g \circ f$ is μ-measurable.

Guidelines:

- Why is it sufficient to prove that preimages of open sets are μ-measurable?

Exercise 6.6.
Let μ be a Borel measure on $\mathbb{R}$ and $f\colon [a, b] \to \mathbb{R}$ continuous μ-almost everywhere (i.e. the set of points where f is not continuous is a μ-zero-set). Show that f is μ-measurable.

Guidelines:

- Recall that sets of measure 0 are measurable,
- In view of that, consider the function $g := f|_{[a,b]\setminus N}$. What property does it have?
- Now, how are the preimages of f related to preimages of g?

Exercise 6.7.
Let μ be a Borel measure on $\mathbb{R}$. Show that every monotone function $f\colon [a, b] \to \mathbb{R}$ is μ-measurable.

Guidelines:

- A monotone function has *very* specific preimages of open intervals.
- Exercise 6.3 should be useful here.