Quick recap:

- A extended-real valued function is *measurable* if preimages of extended half intervals (equivalently: open sets, or Borel sets) are measurable. Example: continuous function.
- Measurability is preserved by algebraic operations and taking suprema and infima.
- A non-negative measurable function can be decomposed into simple functions with coefficients as reciprocal of integers. The corresponding measurable sets are determined by a greedy algorithm. (Proof: include each term whenever possible, and use the divergence of the harmonic series to approximate any positive number from below.)
- *Egoroff's theorem:* a sequence of functions on a set of finite measure with a almost everywhere finite pointwise limit, converge almost uniformly, meaning that away from a set of arbitrarily small positive measure, the convergence is uniform. (Proof: quantify the exceptional sets. Check their intersection has zero measure, hence arbitrarily small at some finite index.)
- *Lusin's theorem:* a measurable function is nearly continuous, meaning that it has a continuous restriction on a compact set which has relatively arbitrarily large measure. (Proof: for simple functions one restricts to the complement of the jump set. A general measurable function is approximated pointiwse by simple functions. By Egoroff's theorem this convergence is uniform on a large compact set, preserving continuity.)
- A *simple function* takes at most countably many values.
- The *integral of a simple function* is a countable sum, with infinity minus infinity avoided in the setting.
- The *integral of a measurable function* is defined when lower and upper integrals (supremum and infimum of integrals of simple functions below and above) agree, called *integrable*
- A measurable function is *summable* if it has a finite integral.

Exercise 7.1.
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with continuous derivative. Show the following inequality:

$$\dim_{\mathcal{H}} f(A) \leq \dim_{\mathcal{H}} A,$$

for all subsets $A \subset \mathbb{R}$.

Guideline:

- The Hausdorff measure is σ-finite.
- A continuously differentiable function is Lipschitz on each bounded set.
- Compute any higher dimensional Hausdorff measure as 0.

Exercise 7.2.
Let $f_k\colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable functions ($k \in \mathbb{N}$). Let:

$$\mathcal{L}^n(\{x \mid \|f_k(x) - f_{k+1}(x)\| > 2^{-k}\}) < 2^{-k}$$

for all $k \in \mathbb{N}$. Show: The limit $\lim\limits_{k\to\infty} f_k(x)$ exists almost everywhere.

Guideline:

- The intersection of the tail of the family $A_k = \{x \in \mathbb{R}^n \mid \|f_k(x) - f_{k+1}(x)\| \leq 2^{-k}\}$ has arbitrarily (in fact geometrically) small measure.
- On each of such intersection, summing up functions with consecutive indices shows that the sequence of functions is Cauchy.

Exercise 7.3.
Let f be a finite, μ-measurable function, and $(f_k)_{k\in\mathbb{N}}$ a sequence of μ-measurable functions with the following property: Every $(f_{k_j})_{j\in\mathbb{N}}$ contains a subsequence, which converges to f in measure μ.

(a) Show that the whole sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in measure μ.

Guideline:

- If not, there is a subsequence without a converging subsubsequence.

(b) Show that the analogous statement from a) is not true, if we assume only pointwise convergence μ-almost everywhere.

Guideline:

- Consider the sequence in Remark 2.4.3, 2), of the Lecture Notes.
- The sequence is not convergent at any given point (Theorem 2.4.2).
- The sequence has a subsequence converging pointwise to zero: For any dyadic length (negative power of 2), pick the interval that supports the given subsequence infinitely often. The subsubsequence supported by these intervals converges.

Exercise 7.4.
Counterexample to $\delta = 0$ in Lusin's Theorem: Find an example of a sequence of $\mathcal{L}^1$-measurable functions $f_k : [0,1] \to \mathbb{R}$ such that for all $M \subset [0,1]$ $\mathcal{L}^1$-measurable sets with $\mathcal{L}^1(M) = 1$, the restriction $f|_M : M \to \mathbb{R}$ is not continuous in every point of M.

Hint: You may use that there exists a Lebesgue-measurable subset $A \subset [0,1]$, such that for all non-empty, open subsets $U \subset [0,1]$, we have:

$$\mathcal{L}^1(U \cap A) \cdot \mathcal{L}^1(U \cap A^c) > 0.$$

Such a set A can be constructed using the Cantor set.

Guideline:

- Take the characteristic function of A as in the hint.
- Around every interior point, pick a sequence of open intervals of arbitrarily small length and use them as U.
- By measurability, there exist two sequences of points, one contained in A and the other in A^c, converging to the same point, contradicting continuity.
- Extra: To construct A use the following procedure.
  1. Put a fat Cantor set (middle $1/4$ gets removed instead) on $[0, 1]$. It has Lebesgue measure $1/2$.
  2. Put a scaled copy of the fat Cantor set in each remaining (removed) interval.
  3. Put a scaled copy of the fat Cantor set in each remaining interval after Step 2.
  4. Repeat indefinitely.

  A is the union of all sets being put in the even steps.

Exercise 7.5.
Counterexample to $\delta = 0$ in Egoroff's Theorem: Find an example of a sequence of $\mathcal{L}^1$-measurable functions $f_k : [0, 1] \to \overline{\mathbb{R}}$, which converges almost everywhere pointwise to the function f, but for every compact $F \subset [0, 1]$ with $\mathcal{L}^1(F) = \mathcal{L}^1([0, 1])$ the convergence on F is not uniformly.

Guideline:

- The monomials on the unit interval indexed by the exponent converges almost everywhere pointwise to zero.
- The convergence is not uniform in any neighborhood of 1.

Exercise 7.6.
Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-measurable function with

$$f(x + y) = f(x) + f(y)$$

(a) Show that f is continuous.
Hint: Use Lusin's Theorem to show that f is continuous at $x = 0$.
Guideline:

- Find $f(0)$ by putting $x = y = 0$.

- By Lusin's Theorem, f is uniformly continuous on a large compact subset F of the unit interval.

- F intersects with its translation by any small number, since their union would have too large a measure if they were disjoint.

- The uniform continuity at such intersection is "shifted to the origin" using the given functional equation.

(b) Show that

$$f(x) = x \cdot f(1).$$

Guideline:

- Put y as integer multiples of x.

- The conclusion holds when x is rational, by the previous step.

- Take a limit using the density of the rationals and the continuity from part (a).

Exercise 7.7.
In this exercise, we construct a set which is Lebesgue-measurable, but not Borel and use the construction to give a counterexample of a continuous $G : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

(a) Let $h : [0,1] \to [0,1]$ be the Cantor function as seen in Ex. 4.3. Define $g : [0,1] \to [0,2]$ by $g(x) := h(x) + x$. Show that g is strictly monotone and a homeomorphism.

Guideline:

- A sum of strictly increasing and monotonically increasing functions is strictly increasing.

- An open interval in $[0,1]$ is homeomorphic under g to an open interval in $[0,2]$. Take a countable union for general open sets.

(b) Denote by $C \subset [0,1]$ the Cantor set. Show that $\mathcal{L}^1(g(C)) = 1$.

Hint: Use the natural decomposition of $[0,1] \setminus C$ to deduce the result.

Guideline:

- Relate the measure of $g(C)$ to the sum of measures of the image of removed intervals under g.

- The Cantor function h is constant on each removed interval.

(c) Use Ex. 4.5.a) to find a non-measurable subset $E \subset g(C)$. Finally, define $A := g^{-1}(E)$. Show that A is a Lebesgue zero set and thus Lebesgue measurable.

Guideline:

- Show that A is contained in C, hence null and measurable.

(d) Show that A is not a Borel set.
Hint: Otherwise, the preimage of A with respect to continuous maps would necessarily be Borel and thus Lebesgue measurable as well.

Guideline:

- Use the hint with the continuous map g^{-1} to conclude that E is Borel, a contradiction.

(e) Find appropriate F, G as outlined above using the sets and functions introduced in the previous subtasks such that $F \circ G$ is not Lebesgue measurable.

Guideline:

- We want E to be the preimage of a Borel set under the composition.
- Rearrange to see that we can take F as the characteristic function of A and G as g^{-1}, and that Borel set is the closed set $\{1\}$.

Exercise 7.8.
Take a Radon measure μ on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Consider a function $f : \Omega \to \mathbb{R}$ which is 0 μ-a.e. Show that f is summable with $\int_\Omega f d\mu = 0$.

Guideline:

- f is measurable because the preimage of a half interval is either a null set or a complement of a null set.
- Compute lower and upper integrals by bounding f by 0 almost everywhere.

Exercise 7.9.
Let $f, g : \Omega \to \overline{\mathbb{R}}$ be μ-integrable and assume it holds:

$$\int_A f d\mu \leq \int_A g d\mu,$$

for all $A \subset \Omega$ μ-measurable. Show that $f \leq g$ μ-almost everywhere. Moreover, conclude that if:

$$\int_A f d\mu = \int_A g d\mu,$$

for all $A \subset \Omega$ μ-measurable, then $f = g$ μ-almost everywhere.

Guideline:

- Put A_n as the set where opposite inequality holds with an extra space of $1/n$.
- Close the inequality and deduce A_n is a null set. Take the limit.

Exercise 7.10.
Denote by μ the Lebesgue measure on $\mathbb{R}$. Find examples for the following statements:

(a) $f_n \to 0$ uniformly, but not $\int |f_n| d\mu \to 0$.

Guideline:

- Look at characteristic functions so that the integral represents the mass (area) of a rectangle.
- Spread the mass with small height and large width.

(b) $f_n \to 0$ pointwise and in measure, but neither $f_n \to 0$ uniformly nor $\int |f_n| d\mu \to 0$.

Guideline:

- Consider characteristic functions again.
- Since uniform convergence is not needed, concentrate the mass by choosing large height and small width.

(c) $f_n \to 0$ pointwise, but not in measure.

Guideline:

- Just look at characteristic functions.
- If we translate the mass to infinity, pointwise convergence is obtained while the measure of the non-converging set is preserved.