In this exercise sheet it is useful to review the results in the notes from Chapter 3 (up until Section 3.4). You will be going through the definition of simple functions up to convergence theorems and differences between Lebesgue and Riemann integrals.

Exercise 8.1.
In this exercise, we prove the linearity, monotonicity and well-definedness of the integral as defined for simple functions, see Def. 3.1.2 and Def. 3.1.3 in the lecture notes. These results are essential to derive the corresponding properties of the general integral.

(a) Let f, g be two μ-measurable simple functions with values $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$, see Def. 3.1.1. in the lecture notes. Show that there exist μ-measurable, disjoint sets $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, such that:

$$f = \sum_n a_n \cdot \chi_{A_n}, \quad g = \sum_n b_n \cdot \chi_{B_n},$$

and prove that the sets and values can be chosen in such a way that $A_n = B_n$ holds for all $n \in \mathbb{N}$.

Guideline: Consider the sets $f^{-1}(\{a_n\})$ and $g^{-1}(\{b_n\})$. Once you can write down A_n and B_n, what can you say about the pairwise intersections?

(b) Show that if $f = \sum_n a_n \cdot \chi_{A_n}$ where $\{a_n\} \subset \overline{\mathbb{R}}$ is a sequence of values (not necessarily different from each other) and $\{A_n\}$ a sequence of pairwise disjoint, μ-measurable subsets. Prove:

$$\int f d\mu = \sum_n a_n \mu(A_n).$$

Guideline: Use Definition 3.1.2/3.1.3.

(c) Let f, g be μ-measurable simple functions such that $f \leq g$ pointwise. It then holds:

$$\int f d\mu \leq \int g d\mu$$

Guideline: By (a) you can assume that you have a decomposition such that $A_n = B_n$ where $f = \sum_n a_n \cdot \chi_{A_n}, \quad g = \sum_n b_n \cdot \chi_{B_n}$.

(d) Assume f, g are μ-measurable simple functions and $a, b \in \mathbb{R}$. Show that $af + bg$ is a μ-measurable simple function and:

$$\int af + bg d\mu = a \cdot \int f d\mu + b \cdot \int g d\mu.$$

Guideline: Same hint as for (c).

(e) Let f be a μ-measurable simple function. Prove:

$$\underline{\int} f d\mu = \overline{\int} f d\mu = \int f d\mu,$$

where the last integral is understood in the sense of integrals for simple functions, see Def 3.1.2/3.1.3 in the lecture notes.

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Exercise 8.2.
Proof the following Theorem: Let $f : \Omega \to \overline{\mathbb{R}}$ be a μ-summable function and

$$|\int_\Omega f \; d\mu| = \int_\Omega |f| \; d\mu \; .$$

Then either $f \geq 0$ or $f \leq 0$ almost everywhere on Ω.

Guideline: When having f map into $\mathbb{R}$, it is sometimes useful to split f into $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Exercise 8.3.
Let $f :]0, 1[\to \mathbb{R}$ be summable. Show that $x^k f$ ist summable as well for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_0^1 x^k f(x) \; dx = 0 \; .$$

Guideline: Use the fact that $x_k \leq 1$ and you want to be able to apply DCT for $x^k f(x)$. Thus, it can be useful to consider the hint in the above question in order to get an integrable majorant.

Exercise 8.4.
Find an example of a continuous, bounded function $f : [0, \infty[\to \mathbb{R}$ with the asymptotic property $\lim_{x\to\infty} f(x) = 0$, such that

$$\int_0^\infty |f(x)|^p dx = \infty \; ,$$

for all $p > 0$.

Guideline: We need a bit of real analysis for this question. e^{-x} satisfies the first property, however the integral will be finite for all $p > 0$. So, this function provides too much decay. Another way of saying this is that e^x has exponential growth. One can test $1/(1+x)$ but again it gives too much decay (for any $p >= 2$ the integral is finite). Is there any function of the form $f(x) = 1/P_r(x)$, for $P_r(x) = (1+x)^r$, that satisfies the required properties? If not, which function has less growth than any $P_r(x)$?

Exercise 8.5.

(a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions on a measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^{\infty} f_k(x)$ converges almost everywhere, if

$$\sum_{k=1}^{\infty} \int_{\Omega} |f_k| dx < \infty \ .$$

Guideline: Monotone convergence.

(b) Let $\{r_k\}$ be an ordering of $\mathbb{Q} \cap [0,1]$; $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$, such that $\sum_{k=1}^{\infty} a_k$ is absolute convergent. Show that then $\sum_{k=1}^{\infty} a_k |x - r_k|^{-1/2}$ is absolute convergent for almost every $x \in [0,1]$.

Guideline: Define $f_k(x) = a_k |x - r_k|^{-1/2}$ and use (a).

Exercise 8.6.

Find an example of a function which is not Lebesgue-summable, such that its improper Riemann integral exists and is finite.

Guideline: Either provide a "well known function", or build example. If you want to build an example, consider the following questions: Which series do you know that converge but do not absolutely converge? How do you relate this to constructing simple functions?

An interesting question which you should ask yourselves is why does there exists such an example in view of Proposition 3.2.2. The critical point is that in Proposition 3.2.2 f is a bounded function defined on a compact domain.