Exercise 9.1.
Let λ be the Lebesgue measure on $\mathbb{R}$ and f a non-negative, summable function on $(\mathbb{R}, \lambda)$. Show that the following equality of Lebesgue integrals holds:

$$\int_{\mathbb{R}} f d\lambda = \int_0^{+\infty} \lambda(\{f > s\}) ds.$$

Guidelines:

- First of all, take the hint seriously and draw pictures! It will make many parts of the proof more transparent.
- Let g be the simple function from the hint. Its image is $\{a_n\}$ and you may assume that $a_i < a_{i+1}$.
- Now you have to interpret the right hand side of the identity you want to prove. For that, consider the sets $\{g > a_i\}$. What follows is a somewhat nasty (but manageable using pictures) double summation which can be reduced to the definition of the Lebesgue integral for a simple function.
- For an arbitrary f, observe first that if g is a simple function, then $s \to \lambda(\{g > s\})$ is again a simple function.

Exercise 9.2.
Take a Radon measure μ on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Let $f, f_1, f_2, ...f_n...$ be non-negative summable functions such that $f_n \to f$ μ-a.e. and $\lim_n \int_\Omega f_n d\mu = \int_\Omega f d\mu$. Take a measurable set $E \subset \Omega$; show that

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Guidelines:

- Prove first that: $\int_E f d\mu = \liminf_n \int_E f_n d\mu$ by using the hint and considering the functions $f\chi_E, f_1\chi_E, ..., f_n\chi_E, ...$
- To prove $\int_E f d\mu = \limsup_n \int_E f_n d\mu$ consider the subsequence $\{g_n\}$ which converges to the $\limsup$ on the righthand side. Now you are again in the situation of the first part of the proof.

Exercise 9.3.
Let f_n be defined by $(n \in \mathbb{N})$:

$$f_n(x) = \frac{n\sqrt{x}}{1 + n^2x^2}, \quad x \in [0, 1].$$

Prove that:

(a) $f_n(x) \leq \frac{1}{\sqrt{x}}$ on $(0,1]$ for all $n \geq 1$;

(b) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Guidelines:

- The most common tool for computing limits of integrals is the Lebesgue Dominated Convergence Theorem. If you can find any function that dominates all the ones from the sequence, this ensures the convergence. What it converges to pointwise is usually easy to compute.

Exercise 9.4.
Show the following result:

Let $\mathbb{R}^n$ be endowed with a Radon measure μ. Take a sequence $\{f_n\}$ of μ-measurable functions on $\mathbb{R}^n$ such that $f_n \to f$ in measure for a μ-measurable f. Suppose that for any $n \in \mathbb{N}$ there is a summable function g_n on $\mathbb{R}^n$ such that $|f_n(x)| \leq |g_n(x)|$ μ-a.e. and assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |g_n - g| d\mu = 0$$

for a summable function g. Then the following holds:

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |f_n - f| d\mu = 0.$$

Guidelines:

- Integrability of g allows us to approximate $\int_{\mathbb{R}^n} |g| d\mu$ arbitrarily well by integrating $|g|$ only over a ball of radius R, for R sufficiently large.
- The mentioned result from the lecture shows that, for this R, $\int_{B_R(0)} |f_n - f| d\mu$ is arbitrarily small for large enough n.
- Some more work and these two approximations then show that $\int_{\mathbb{R}^n} |f_n - f| d\mu$ is arbitrarily small for large enough n.

Exercise 9.5.
Alternative proofs of Young's inequality. Young's inequality states that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$. Prove the inequality by taking the logarithm of both sides.

Guidelines:

- Recall Jensen's Inequality from Analysis.

Exercise 9.6.
Let μ be a Radon measure on $\mathbb{R}^n$ and $f : \Omega \to \overline{\mathbb{R}}$ a μ-summable function. For $A \subset \Omega$ μ-measurable, define:

$$\nu(A) = \int_A f d\mu.$$

For general subsets, define ν as in (1.2.32) in the lecture notes. Show that for $f \geq 0$ μ-a.e. ν is a Radon measure on $\mathbb{R}^n$.

Guidelines:

- Check first if ν satisfies the criteria for a Carathéodory-Hahn extension.

- Use the Carathéodory criterion of measurability to prove that it is Borel.

- For Borel regularity, use the fact that μ is Borel regular to produce, for A μ-measurable, a Borel set B, such that $A \subset B$ and $\mu(A) = \mu(B)$. Compare $\nu(B)$ and $\nu(A)$ from defintion of ν.