Topology with many open Sets

Gabriel Dettling

FS 2020

Let $X = \{1, ..., n\}$, and let $\mathcal{O}$ be a topology on X with $|\mathcal{O}| > \frac{3}{4}2^n$. For any $i, j \in X$ with $i \neq j$, consider the set

$$\mathcal{S}_{ij} := \{A \subseteq X |\, i \in A,\, j \notin A\}$$

and note that $|\mathcal{S}_{ij}| = \frac{1}{4}2^n$. If we had $\mathcal{S}_{ij} \cap \mathcal{O} = \emptyset$, that would imply

$$|\mathcal{P}(X)| \geq |\mathcal{O}| + |\mathcal{S}_{ij}| > 2^n$$

which is a contradiction to $|\mathcal{P}(X)| = 2^n$.

Therefore we can pick $\Omega_{ij} \in (\mathcal{S}_{ij} \cap \mathcal{O})$ for all $i \neq j$. For any $i \in X$, we have

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{n} \Omega_{ij} = \{i\} \in \mathcal{O}$$

(finite intersection of open sets is open). As every (nonempty) subset A of X can be written as $A = \bigcup_{i \in A}\{i\}$, it follows that $\mathcal{O}$ is the discrete topology.

This shows that there can be no topology $\mathcal{O}'$ on X with $\frac{3}{4}2^n < |\mathcal{O}'| < 2^n$.