Topology with many open sets

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Let $X = \{1, 2, 3, 4, ..., n\}$.

Claim: The only topology τ on X with $|\tau| > 3 * 2^{n-2}$ is the discrete topology.

Def: We call a binary relation $\leq$ on a set P a preorder if
i)$\forall a \in P : a \leq a$
ii)$\forall a, b, c \in P : (a \leq b \cup b \leq c \Rightarrow a \leq c)$

Lemma: There exist a one-to-one correspondence between topologies and preorders on X:
a) For any preoder $\leq$ on X,
$\tau := \{A \subset X \mid \forall x \in A : (\forall y \in X : x \leq y \Rightarrow y \in A)\}$ defines a topology.
b) For any topology τ on X, x $\leq$ y if and only if x $\in \overline{\{y\}}$ defines a preoder.

Proof of the Lemma.
a) Note that
i) $\emptyset, X \in \tau$
ii) $\forall A, B \in \tau : A \cap B \in \tau$ as
$\forall x \in A \cap B : \forall y \in X : (x \leq y \Rightarrow y \in A \wedge y \in B \Rightarrow y \in A \cap B)$
iii) $\forall A, B \in \tau : A \cup B \in \tau$ as
$\forall x \in A \cup B : \forall y \in X : (x \leq y \Rightarrow y \in A \vee y \in B \Rightarrow y \in A \cup B)$

Since X is finite, this implies that τ is a topology.

b) Note that
i) $\forall x \in X : x \in \overline{\{x\}} \Rightarrow x \leq x$ (reflexive)
ii) $\forall x, y \in X : (x \leq y, y \leq z \Rightarrow x \in \overline{\{y\}}, y \in \overline{\{z\}} \Rightarrow x \in \overline{\{y\}} \subset \overline{\{z\}})$ (transitive)

Therefore, $\leq$ defines a preorder on X.

Proof of the claim.
According to the Lemma, it suffices to consider the topologies induced by preorders on X. Assume that the preorder $\leq$ is not trivial and let τ denote the topology induced by it.

It holds that $\exists x, y \in X : x \leq y$ and $\forall A \subset X : (x \in A \wedge y \notin A \Rightarrow A \notin \tau)$. Note that there are 2^{n-2} distinct subsets A of X with $x \in A \wedge y \notin A$ and hence $|\tau| \leq 3 * 2^{n-2}$. Since the discrete topology is induced by the trivial preorder, it is the only topology satisfying $|\tau| > 3 * 2^{n-2}$.

q.e.d.