2.10. Claim: $\forall n,k \in \mathbb{N}_*$ there exist k distinct open sets in $\mathbb{R}^n$ with the same boundary.

Proof by constructing such sets directly:

Let $n = 1$.

For $a \in \mathbb{Z}$ I define $I_a := (a, a+1)$.

I_a is open.

$$\mathbb{R} \setminus \mathbb{Z} = \bigcup_{a \in \mathbb{Z}} I_a$$

Define $A_1 := \bigcup_{a \in \mathbb{Z}} I_a = \mathbb{R} \setminus \mathbb{Z}$. A_1 is open.

$\overline{A_1} = \mathbb{R}. \Rightarrow \partial A_1 = \mathbb{Z}$

Now notice that in $\bigcup_{a \in \mathbb{Z}} I_a$, we can leave one intervall out

and still get a set whose boundary is $\mathbb{Z}$.

$\Rightarrow$ Define for $2 \leq i \leq k$ $A_i = \bigcup_{a \in \mathbb{Z} \setminus i} I_a$ A_i is open.

$\overline{\partial A_i} = \mathbb{Z}$

This gives us k distinct open sets whose boundaries are always $\mathbb{Z}$.

If $n > 1$ we simply consider $\tilde{A}_i := A_i \times \mathbb{R}^{n-1}$. The boundaries are then $\mathbb{Z} \times \mathbb{R}^{n-1}$.