# Topology - Exercise Sheet 3 ; Challenge Problem

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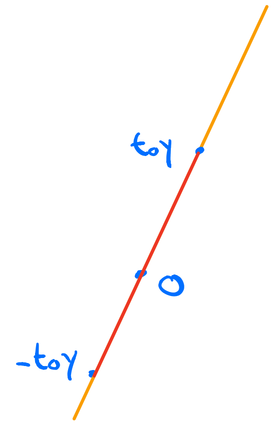
## Exercise 3.10

**Claim** $0 \in G$.

**Proof** Since $G$ is non-empty, there is some $x \in G$. As $G$ is symmetric, $-x \in G$.
As $G$ is closed under addition, $0 = x + (-x) \in G$. $\square$




This also means, that $|G| \geq 1$.

If $|G| = 1$, then $G = \{0\}$ and $G$ is a linear subspace of dim $0$.

Assume $G \subseteq \{ty \mid t \in \mathbb{R}\}$ for some $y \in \mathbb{R}^2 \setminus \{0\}$ and $|G| > 1$.

**Claim** $G = \{ty \mid t \in \mathbb{R}\}$

**Proof** Since $|G| > 1$, there is some $t_0 \in \mathbb{R} \setminus \{0\}$, s.t. $t_0 y \in G$. By symmetry also $-t_0 y \in G$.
By path-connectedness of $G$, also $ty \in G$ for $t \in [-t_0, t_0]$.
Thus, for an arbitrary $\tilde{t} \in \mathbb{R}$, there is an $n \in \mathbb{N}$, s.t. $nt_0 \leq \tilde{t} < (n+1)t_0$.
Especially, $0 \leq \tilde{t} - nt_0 < t_0$, thus $t := \tilde{t} - nt_0 \in [-t_0, t_0]$ and, as $G$ is closed under addition, $\tilde{t}y = (nt_0 + t)y \in G$. $\square$

Thus, in this case, $G$ is a lin. subspace of dim. $1$.

In each other case, there are $x, y \in \mathbb{R}^2$, lin. indep. with $x, y \in G$.

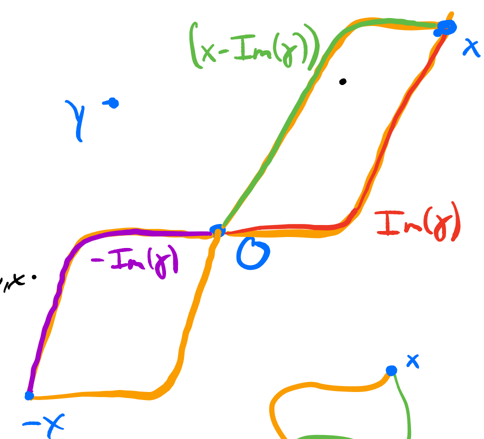
Let $\gamma_{0,x} : [0,1] \to G$ be a path from $0$ to $x$. W.L.o.G. it is injective.
Then also $-(\gamma_{0,x}([0,1])) \subset G$ by symmetry of $G$. But then, also $x - (\gamma_{0,x}([0,1])) \subset G$, by closedness under addition of $G$.

If $\gamma_{0,x}([0,1]) = x - (\gamma_{0,x}([0,1]))$, then we get, that $\operatorname{span}\{x\} \subset G$ as in the claim before.

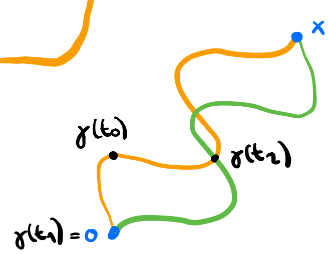
If this is the case, consider $\gamma_{0,y} : [0,1] \to G$, connecting $0$ to $y$.
If again $\gamma_{0,y}([0,1]) = y - (\gamma_{0,y}([0,1]))$, then also $\operatorname{span}\{y\} \subset G$.

But bcs., by asspt. $x, y$ are lin. indep., that'd mean, that $G = \mathbb{R}^2$.

So assume W.L.o.G., $\gamma_{0,x}([0,1]) \neq x - (\gamma_{0,x}([0,1]))$. Write $\gamma = \gamma_{0,x}$.
Then there is some $t_0 \in (0,1)$ s.t. $\gamma(t_0) \notin (x - \gamma([0,1]))$.




Consider
$$t_1 = \inf\{t \in [0,1] : \gamma((t, t_0)) \cap (x - \gamma([0,1])) = \emptyset\}$$
$$t_2 = \sup\{t \in [0,1] : \gamma((t_0, t)) \cap (x - \gamma([0,1])) = \emptyset\}.$$

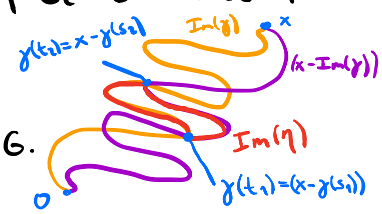



They both exist, as $t_0$ is in both sets, and both sets are naturally bounded.

Both sets are open in $\mathbb{R}$, thus there are $s_1, s_2 \in [0,1]$, W.L.o.G. $s_1 < s_2$, s.t.
$$x - \gamma(s_2) = \gamma(t_1), \quad x - \gamma(s_1) = \gamma(t_2).$$

Define the path $\eta : [t_1, t_2 + s_2 - s_1] \to G;\ t \mapsto \begin{cases} \gamma(t), & \text{if } t_1 \le t \le t_2 \\ x - \gamma(t - t_2 + s_1), & \text{if } t_2 \le t \le t_2 + s_2 - s_1 \end{cases}$.

By construction, $\eta$ is injective, closed, cont. Thus $\eta$ is a Jordan path.




Assume, that there is some $x_0 \in \mathbb{R}^2$, with $x_0 \notin G$. Then also $x_0 \in \mathbb{R}^2 \setminus \eta$, as $\eta \subset G$.

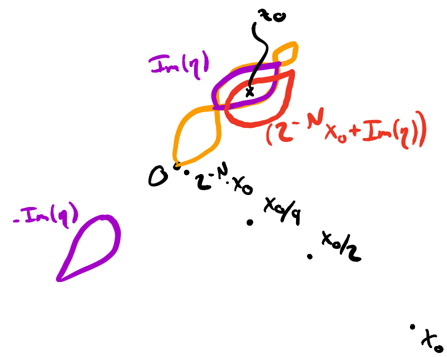
Choose $z_0$ in the bounded component of $\mathbb{R}^2 \setminus \eta$.

As $\mathrm{Im}(\eta)$ is closed $\mathbb{R}^2 \setminus \eta$ is open and thus there is $\varepsilon > 0$, s.t. $B_\varepsilon(z_0) \subseteq \mathbb{R}^2 \setminus \eta$.

Claim $\forall n \in \mathbb{N}_0$: $2^{-n} \cdot x_0 \in G^c$.

Proof For $n = 0$, this is true.
Assume it's true for some $n$. Then, if $2^{-n-1} \cdot x_0$ was in $G$, $2^{-n-1} x_0 + 2^{-n-1} x_0 = 2^{-n} x_0 \in G$, by closedness of $G$ under addition, contradicting the assp.

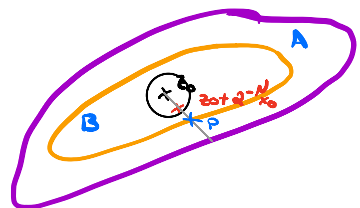
Choose $N \in \mathbb{N}$ big enough, s.t. $|2^{-N} x_0| < \varepsilon$.
Notice, that
$$\left( 2^{-N} x_0 + \eta([0,1]) \right) \subset G^c,$$
bcs. o.w. $2^{-N} x_0 = (2^{-N} x_0 + \eta(t)) - \eta(t) \in G$ for some $t \in [0,1]$.



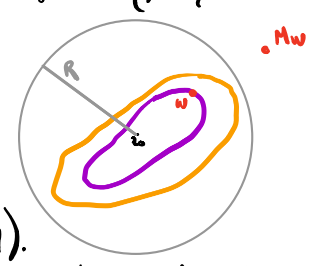

Claim $\exists z \in \mathrm{Im}(\eta) \cap (2^{-N} x_0 + \mathrm{Im}(\eta))$

Proof Assume $\mathrm{Im}(\eta) \cap (2^{-N} x_0 + \mathrm{Im}(\eta)) = \emptyset$. Then $2^{-N} x_0 + \mathrm{Im}(\eta) \subseteq A$, where $A$ is one of the two path-connected components of $\mathbb{R}^2 \setminus \eta$.

First, consider the case, that $A$ is the component with $z_0 \in A$.
Define
$$T := \sup\{ t \in \mathbb{R}_{\ge 0} \mid z_0 + t\, 2^{-N} x_0) \in B \},$$




where $B$ is the path-connected component of $\mathbb{R}^2 \setminus (2^{-N} x_0 + \mathrm{Im}\,\eta)$ with $z_0 + 2^{-N} x_0 \in B$. In part. $z_0 \in B$, as $|2^{-N} x_0| < \varepsilon$ and $B_\varepsilon(z_0 + 2^{-N} x_0) \subseteq B$, and bcs. $\mathrm{Im}(\eta) \cap (2^{-N} x_0 + \mathrm{Im}\,\eta) = \emptyset$ also $B \subseteq A$.
$T$ is well-defined, as $2^{-N} x_0 + \mathrm{Im}(\eta)$ is bounded as it is compact as the continuous image of a compact subset of $\mathbb{R}$.
In particular, $P = T(z_0 + 2^{-N} x_0) \in (2^{-N} x_0 + \mathrm{Im}(\eta))$. $P \in A$, as $\mathrm{Im}(\eta) \cap (2^{-N} x_0 + \mathrm{Im}\,\eta) = \emptyset$
But then
$$(2^{-N} x_0 + \mathrm{Im}(\eta)) - 2^{-N} x_0 \neq \mathrm{Im}(\eta),$$
bcs. $P - 2^{-N} x_0 \in A$ and $A \cap \mathrm{Im}(\eta) = \emptyset$. ↯.




If $z_0 \notin A$, choose $R > 0$ with $2^{-N} x_0 + \mathrm{Im}(\eta) \subseteq B_R(z_0)$ (it's bounded).
Let $w \in \mathrm{Im}(\eta)$ with $w \neq 0$. Then there is $M \in \mathbb{N}$ with $Mw \notin B_R(z_0)$. But then $w, Mw \in G$ but both are in different path-connected components of $\mathbb{R}^2 \setminus (2^{-N} x_0 + \mathrm{Im}(\eta))$, but $G$ is p.c. ↯ □

But this claim shows, that there is $z \in G \cap G^c = \emptyset$. ↯
Thus $G^c = \emptyset$, $G = \mathbb{R}^2$, in part., $G$ is a lin. subspace of $\mathbb{R}^2$ of dim. 2. □

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