Topology

Ex 4.10

Suppose there exists a coverage of $\mathbb{R}^n$ by countable strictly convex compact pairwise disjoint sets $(K_i)_{i\in\mathbb{N}}$. Then since every K_i is strictly convex and compact, $K_i \cap \mathbb{R}\times\{0\}^{n-1}$ is either empty or a closed bounded interval. Hence we get a cover of $\mathbb{R}$ by closed bounded pairwise disjoint intervals, which are in fact the strictly convex compact sets of $\mathbb{R}$. This shows that it's sufficient to show that such cover is not possible for $n=1$.

Suppose we have $\{I_k \mid k\in\mathbb{N}\}$ s.t. $\dot{\bigcup}_{k\in\mathbb{N}} I_k = \mathbb{R}$.
Since all I_k are bounded, we have at least two distinct $J_p, J_q, b_p < a_q$. We write $a=b_p$, $b=a_q$. Notice that there exists infinite $(\tilde{I}_k)_{k\in\mathbb{N}}$ s.t. $(a,b)=\dot{\bigcup}_{k=1}^{\infty}\tilde{I}_k$, since every finite union of $\tilde{I}_k$s would be bounded and closed, hence compact, but (a,b) isn't.

We will now define a function $f:[0,1]\to\mathbb{R}$ as follows:
Let $f(0)=a$, $f(1)=b$, $f(\frac{1}{2})=\frac{a+b}{2}$. Now there exists exactly one Interval $\frac{a+b}{2}\in\tilde{I}=[\tilde{a},\tilde{b}]$. Then $(a,b)\setminus\tilde{I}$ consists of exactly two disjoint open intervals $(a,\tilde{a}),(\tilde{b},b)$. One can now repeat this step for every open interval, i.e. $f(\frac{1}{4}):=\frac{a+\tilde{a}}{2}$, $f(\frac{3}{4}):=\frac{\tilde{b}+b}{2}$ and then repeat this with the remaining four open sets. Inductively, one can now define $f(x)$ for every $x\in[0,1]$ which can be written as $\frac{k}{2^n}$ by the construction above. Those are also exactly the x for which there exists a finite representation $0.x_1x_2\ldots x_n$ of x with base 2. Note that for those x, f is strictly increasing.

Now consider an abritary $x\in[0,1]$. We then consider the unique representation $0.x_1x_2\ldots$ of x with base 2, s.t. $\nexists N: \forall n\geq N: x_n=1$. We now consider the sequence $(y_i)_{i\in\mathbb{N}}$, by $y_i := 0.x_1\ldots x_i$. Now let $O_{n_1}\ldots O_{n_{2^n}}$ be the open sets as above in the construction, s.t. $f(\frac{k}{2^n})$ is the midpoint of O_{n_k}. By construction, for all k, the length of O_{n_k} is at most $\frac{1}{2^n}$. Now for any x, for which $f(x)$ is still not defined, i.e. all x with an infinite representation, we have that $\forall N\ \exists n\geq N$ s.t. $x_n=1$. Then (y_n) is the midpoint of one O_{n_k}. Then $\forall m\geq n$ we have $y_m\in O_{n_k}\subseteq B_{y_n}(\frac{1}{2^n})$, hence y_n has to be a Cauchy sequence, meaning there exists a limit. With this we can define $f(x)=\lim_{n\to\infty} y_n$ for any other x.
Furthermore by construction, we have that if $x_n=1$, y_n and y_{n-1} have to lie in distinct Intervals. This especially implies that there is no n, s.t. y_n and $f(x)$ lie in the same Interval.

Now consider $\Omega = \{x \in [0,1] \mid x$ has a non finite representation$\}$. Let $x < z \in \Omega$ be arbitrary. Let $n_0 := \min\{n \mid x_n \neq z_n\}$. Then $x_{n_0} = 0$, $z_{n_0} = 1$. By assumption, we know that $\exists n_1, n_2 > n_0$ s.t. $x_{n_1} = 0$, $z_{n_2} = 1$. Then $\forall n > \max(n_1, n_2)$ we know since f for x with finite represantation is strictly increasing that $0,x_1 \dots x_n < 0,z_1 \dots z_{n_0} < 0,z_1 \dots z_n$
$\Rightarrow f(0,x_1 \dots x_n) < f(0,z_1 \dots z_{n_0}) < f(0,z_1 \dots z_n)$, hence for $n \to \infty$ we have
$f(x) = f(0,x_1 \dots) \leq f(0,z_1 \dots z_{n_0}) < f(0,z_1 \dots z_{n_2}) < f(0,z_1 \dots) = f(z)$. Hence $f|_\Omega$ is injective.
Furthermore we know that since $f(x) < f(0,z_1 \dots z_{n_2}) < f(z)$, and that $f(0,z_1 \dots z_2)$ and $f(z)$ have to lie in different intervals, $f(x)$ and $f(z)$ have to lie in different intervals.
With this, we can define an injective function $\varphi : \Omega \hookrightarrow \{I_k \mid k \in \mathbb{N}\}$. Since $(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \subseteq \Omega$, we know that Ω is uncountable, which implies that there cannot just be countable many intervals.

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