Topology - Exercise Sheet 5; Challenge Problem

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I^I is compact by Tychonoff thm., as $I=[0,1]$ is closed, bounded in $\mathbb{R}$ and thus comp.
We do now show, that I^I is not sequentially compact.

Let $(x_n)_{n\in\mathbb{N}} \subset I^I$ be a sequence, where $\forall n\in\mathbb{N}$: $x_n = (x_n^{(i)})_{i\in I}$.

Claim 1: $(x_n)_n$ converges to $x=(x^{(i)})_i \in I^I$ in the I^I prod. top. $\iff$ $\forall i\in I$: $x_n^{(i)}$ converges to $x^{(i)}$ in $I\subset\mathbb{R}$.

Proof: "$\Rightarrow$" Let $i_0\in I$. Let $\varepsilon>0$. Then

$$\mathcal{U} := \Big(\prod_{i\in I\setminus\{i_0\}} I\Big) \times B_\varepsilon(x^{(i_0)}) \subset I^I$$

is open and $x\in\mathcal{U}$. Thus, there is $N\in\mathbb{N}$, s.t. $x_n\in\mathcal{U}$ for all $n\geq N$.

But then, also $x_n^{(i_0)}\in B_\varepsilon(x^{(i_0)})$ for all $n\geq N$, but each such n thus fulfills

$$|x_n^{(i_0)} - x^{(i_0)}| < \varepsilon.$$

Since $\varepsilon>0$ was arbitrary, $x_n^{(i_0)} \to x^{(i_0)}$ as $n\to\infty$.

"$\Leftarrow$" Let $\mathcal{U}\subset I^I$ be open, s.t. $x\in\mathcal{U}$. By def. of the product topology, there is $h\in\mathbb{N}$, $\mathcal{U}_{i_1},\dots,\mathcal{U}_{i_h}\subseteq I$ open, s.t.

$$\mathcal{U} = \Big(\prod_{i\in I\setminus\{i_1,\dots,i_h\}} I\Big)\times\Big(\prod_{j=1}^{h}\mathcal{U}_{i_j}\Big)$$

By openness of $\mathcal{U}_{i_j}$, there is $\varepsilon_j>0$, s.t. $B_{\varepsilon_j}(x^{(i_j)})\subset\mathcal{U}_{i_j}$ for $j\in\{1,\dots,h\}$.
For $j\in\{1,\dots,h\}$, by assspt.

$$\exists N_j\ \forall n\geq N_j:\ x_n^{(i_j)}\in B_{\varepsilon_j}(x^{(i_j)}).$$

Define $N:=\max\{N_1,\dots,N_h\}$. Then

$$\forall j\in\{1,\dots,h\}\ \forall n\geq N:\ x_n^{(i_j)}\in B_{\varepsilon_j}(x^{(i_j)})\subset\mathcal{U}_{i_j}.$$

In part.,

$$\forall n\geq N:\ x_n\in\mathcal{U},$$

thus $x_n\to x$ $(n\to\infty)$. $\square$

Consider $\Phi: I\to\{0,1\}^{\mathbb{N}}$ which maps every $i\in I$ to it's corresponding binary expansion (the seq. $(b_1,b_2,b_3,\dots)$, s.t. $i=\sum_{j=1}^{\infty}\frac{b_j}{2^j}$).

Consider the seq. $(x_n)_n\subset I^I$, where $x_n^{(i)} := (\Phi(i))_n$ for all $n\in\mathbb{N}$, $i\in I$, where with $(\Phi(i))_n$, we denote the n'th component of $\Phi(i)$.

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Claim 2: $(x_n)_n$ has no convergent subsequence.

Proof: Let $(x_{n_k})_k \subset (x_n)_n$ be any subsequence. Then, for $i_0 = \sum_{\ell=1}^{\infty} \frac{b_\ell}{2^\ell}$, where

$$b_\ell = \begin{cases} 0, & \text{if } (\exists k \in 2\mathbb{N}: \ell = n_k) \text{ or } (\forall k \in \mathbb{N}: \ell \neq n_k) \\ 1, & \text{if } \exists k \in \mathbb{N} \setminus (2\mathbb{N}): \ell = n_k \end{cases}$$

, we have

$((\Phi(i_0))_{n_k})_k = (1, 0, 1, 0, 1, 0, \dots)$ and thus

$\lim_{k \to \infty} x_{n_k}^{(i_0)}$ doesn't exist $\xrightarrow{\text{Claim 2}}$ $\lim_{k \to \infty} x_{n_k}$ doesn't exist.

This shows, that every subsequence of $(x_n)_n$ is divergent. $\square$

By claim 2, there is a sequence in I^I, which has no convergent subsequence.

In part., I^I is <u>not</u> seq. compact.