Proposition Let (X, d) be a compact metric space. Let $f : X \to X$ be an expansive map, then f is an isometry.

Proof: Let (X, d) be a compact metric space, f an expansive map. This implies that X is also sequentially compact. Fix $x, y \in X$. Define two sequences $x_n = f^{n-1}(x)$ and $y_n = f^{n-1}(y)$. We now use the expansive property of the map to write

$$d(f(x), f(y)) \leq ... \leq d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$

and the triangle inequality in the last step. Because X is sequentially compact, we can find for each sequence a converging subsequence. Denote them by x_{n_k}, y_{n_k}. We use convergence of x_{n_k}, y_{n_k} to write out two properties $\forall \epsilon > 0\ \exists n_k$ such that $d(x_{n_k}, x_{n_{k+1}}) < \epsilon$ and $d(y_{n_k}, y_{n_{k+1}}) < \epsilon$. Let $m_k = n_{k+1} - n_k$. Then, because of the expansive property, we can write again: $d(x_{m_k}, x) \leq d(x_{n_{k+1}}, x_{n_k}) < \epsilon$ and $d(y_{m_k}, y) \leq d(y_{n_{k+1}}, y_{n_k}) < \epsilon$. We reinsert into the inequality chain from above and conclude:

$$d(f(x), f(y)) \leq ... \leq d(x_{m_k}, y_{m_k}) \leq d(x_{m_k}, x) + d(x, y) + d(y, y_{m_k}) < \epsilon + \epsilon + d(x, y)$$

. By taking the limit with $\epsilon \to 0$, we see that $d(f(x), f(y)) \leq d(x, y))$. By assumption $d(x, y) \leq d(f(x), f(y))$. So $d(x, y) = d(f(x), f(y))$. $\square$