Topology Challenge Problem 8

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Consider $Y := \bigcup_{r\in\mathbb{Q}\cap(0,1]} \{r\}\times(0,r]$ and $Z := \bigcup_{r\in\mathbb{Q}\cap(-1,0]} (0,1+r]\times\{r\}$. Let $Y_k := Y+(k,k)$ and $Z_k := Z+(k,k)$ for $k\in\mathbb{Z}$ and $X := \bigcup_{k\in\mathbb{Z}}(Y_k \cup Z_k)$, where we can see that this indeed is a disjoint union since $Y_k \subseteq (k,k+1]\times(k,k+1]$ and $Z_k \subseteq (k,k+1]\times(k-1,k]$.

A picture would have helped a lot.

Let $H: X\times[0,1]\to X$ be defined as follows:

Here should be y \in (0, r]

For $x\in Y_k$, we have an unique $r\in\mathbb{Q}\cap(0,1]$, s.t. $x=\binom{k+r}{k+y}$ with $y\in(k,k+r]$. For $r+y<1$ we define

$$H(x,t) := \begin{cases} x-\binom{0}{t} & \text{for } t\in[0,y) \\ x-\binom{t-y}{y} & \text{for } t\in[y,y+r) \\ x-\binom{r}{t-r} & \text{for } t\in[y+r,1] \end{cases} \text{ and for } r+y\geq 1,\ H(x,t) := \begin{cases} x-\binom{0}{t} & \text{for } t\in[0,y) \\ x-\binom{t-y}{y} & \text{for } t\in[y,1]. \end{cases}$$

Then H is well defined since for $t\in[0,y)$, $H(x,t)\in\{k+r\}\times(k,k+r]\subseteq Y_k$, for $t\in[y,y+r)$, $H(x,t)\in(k,k+1]\times\{k\}\in Z_k$ and eventually for $t\in[y+r,1]$, $H(x,t)\in\{k\}\times(k-1,k]\subseteq Y_{k-1}$.

Analogeously, for $x\in Z_k$ and $r\in\mathbb{Q}\cap(-1,0]$ s.t. $x=\binom{k+y}{k+r}$ with $y\in(0,1+r]$ for $y+(1+r)<1$ we define

$$H(x,t) := \begin{cases} x-\binom{t}{0} & \text{for } t\in[0,y) \\ x-\binom{y}{t-y} & \text{for } t\in[y,y+1+r) \\ x-\binom{t-(1+r)}{1+r} & \text{for } t\in[y+1+r,1] \end{cases} \text{ and for } y+(1+r)\geq 1,\ H(x,t) := \begin{cases} x-\binom{t}{0} & \text{for } t\in[0,y) \\ x-\binom{y}{t-y} & \text{for } t\in[y,y+1+r). \end{cases}$$

For the same reason we can again see that $H(x,t)$ is well defined.

One can quickly check that $H(x,t)$ is continuous in t for any x. To see that H is also continuous in x, we first define $\Phi\left(\binom{a}{b},\binom{c}{d}\right) := |a-c|+|b-d|$. Consider any $x,y\in X$, $t\in[0,1]$. By construction, for any Δt small enough, $H(x,t+\Delta t)$ is either $H(x,t)-\binom{0}{\Delta t}$ or $H(x,t)-\binom{\Delta t}{0}$. Thus Φ could only increase, if there is Δt, s.t. for $H(x,t)=\binom{a}{b}$, $H(y,t)=\binom{c}{d}$ we have $a<c$, $b<d$, $H(x,s)=\binom{a-\Delta t}{b}$ and $H(y,s)=\binom{c}{d-\Delta t}$ or viceversa. In this case there are k,j, s.t. $H(x,t)\in Z_k$ and $H(y,t)\in Y_j$, but this cannot hold, under the statement above. Thus $\Phi(H(x,t),H(y,t))$ cannot increase.

Now for any x,y, we have $\Phi(x,y)\geq|x-y|$ by the triangle equation, but also $2|x-y|\geq\Phi(x,y)$. Thus $\forall\varepsilon>0$, we have $\forall t$, $\forall x,y\in X$ s.t. $|x-y|<\frac{\varepsilon}{2}$, that $\varepsilon>\Phi(x,y)=\Phi(H(x,0),H(y,0))$ $\geq\Phi(H(x,t),H(y,t))\geq|H(x,t),H(y,t)|$. Thus H is continuous.

Consider $W = \bigcup_{k \in \mathbb{Z}} ([k, k+1] \times \{k\} \cup \{k+1\} \times [k, k+1])$. Note that $W \subseteq X$ and $H(X, 1) \subseteq W$.

W is homeomorph to $\mathbb{R}$, you can for example project W onto $\langle \binom{1}{1} \rangle$ as homeomorphism.

$\mathbb{R}$ is contractible, so W also have to be contractible. Thus there is $G: W \times [0,1] \to W$ s.t.

G is continuous, $G(\cdot, 0) = id_W$, $G(\cdot, 1) = (x \mapsto x_0)$ for a $x_0 \in W$.

If we define $F: X \times [0,1] \to X, (x,t) \mapsto \begin{cases} H(x, 2t) & \text{for } t \in [0, \frac{1}{2}] \\ G(H(x, 1), 2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$, F is well defined and continuous. Furthermore, $\forall x: F(x, 0) = H(x, 0) = x$ and $F(x, 1) = G(H(x, 1), 1) = x_0$, thus X is contractible.

Suppose X is deformation retractable to $x_0 \in X$. Then for any neighbourhood $x_0 \in U \subseteq X$, there is a neighbourhood $x_0 \in V \subseteq U$, s.t. the inclusion $i: V \hookrightarrow U$ is homotopic in U to $c_{x_0}: V \to U, x \mapsto x_0$.

W.l.o.g. $x_0 = (x, y) \in Y_0$, the other cases are analogeous. Thus consider $U := (x - \frac{y}{2}, x + \frac{y}{2}) \times (\frac{y}{2}, \frac{3y}{2}) \cap X$.

Since $\mathbb{Q}$ is dense in $[0,1]$, for any neighbourhood $V \subseteq U$ of (x,y), $\exists (r,s) \in V \cap \{x\} \times (0, 1-x]$. Now there cannot be a path from (r,s) to (x,y) in U. But if a Homotopy $K: V \times [0,1] \to U$ exists, s.t. $K(\cdot, 0) = i$ and $K(\cdot, 1) = c_0$, then $K((r,s), \cdot)$ would be a continous path from (r,s) to (x,y) in U, thus such K cannot exist. This shows that X is not deformation retractable to any point $x_0 \in X$.