Topology Ruben Skorupinski

9.10 $\gamma := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=1, z=0\}$

$r := \{(x,y,z)\in\mathbb{R}^3 \mid x=y=0\}$

UTS $\mathbb{R}^3\setminus\gamma \underset{\text{homeo}}{\not\cong} \mathbb{R}^3\setminus(\gamma\cup r)$

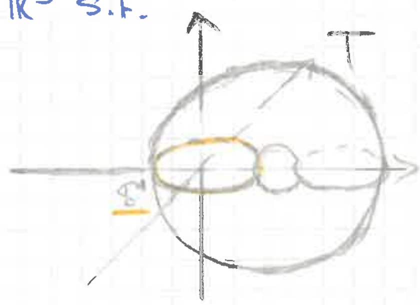
Proof Idea: show $\pi_1(\mathbb{R}^3\setminus\gamma) \underset{\text{iso}}{\not\cong} \pi_1(\mathbb{R}^3\setminus(\gamma\cup r))$

- $\pi_1(\mathbb{R}^3\setminus\gamma)$:

Consider the following sets: $D^2 := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1, z=0\}$

$D_0^2 := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2 < 1, z=0\}$

$T := S^1\times D_0^2$, i.e. an open, filled torus placed in $\mathbb{R}^3$ s.t. D_0^2 is a cross-section.



Set $A := \mathbb{R}^3\setminus D^2$

$B := T$

$\Rightarrow$ A & B are open and path-connected, furthermore their intersection $A\cap B = T\setminus D_0^2$ is also path-connected

$A\cap B$ is contractible since $T\setminus D_0^2 = (S^1\setminus\{x_0\})\times D_0^2$

$\underset{\uparrow\text{stereographic proj.}}{\cong} \mathbb{R}\times D_0^2 \cong (0,1)\times D_0^2$ which is just an open cylinder $\subseteq\mathbb{R}^3$ $\to$ contractible.

$\Rightarrow \pi_1(A\cap B) = \{0\}$ $\Rightarrow$ by comment 2b of the lecture $\pi_1(X) = \pi_1(A) * \pi_1(B)$

$T \underset{\text{homotopic}}{\simeq} S^1$ as D_0^2 is contractible $\Rightarrow \pi_1(B) = \mathbb{Z}$.

$A \simeq \mathbb{R}^3\setminus\{0\} \simeq S^2$ using $\begin{cases} f: S^2 \to \mathbb{R}^3\setminus\{0\}, & f(x)=x \\ g: \mathbb{R}^3\setminus\{0\}\to S^2, & g(x) = \frac{x}{\|x\|}\end{cases}$

then $g\circ f = id_{S^2}$, $(f\circ g)(x) = \frac{x}{\|x\|} \rightsquigarrow$ homotopic to $id_{\mathbb{R}^3}$ via $H(x,t) = tx + \frac{x}{\|x\|}(1-t)$

In the lecture we've proven that $\pi_1(S^2) = \{0\}$

$\Rightarrow \pi_1(\mathbb{R}\setminus\gamma) = \pi_1(B) * \{0\} = \pi_1(B) = \mathbb{Z}$.

- $\pi_1(\mathbb{R}^3\setminus(\gamma\cup r))$:

First note that we can express $\mathbb{R}^3\setminus r$ as $S^1\times\mathbb{R}^2_+$

where we define $\mathbb{R}^2_+$ as $\{(x,z)\in\mathbb{R}^2 \mid x>0\}$

$\Rightarrow$ in a similar manner we find $\mathbb{R}^3\setminus(\gamma\cup r)$ to be equal to $S^1\times P$ where $P := \{(x,z)\in\mathbb{R}^2 \mid x>0, (x,z)\neq(1,0)\}$

$\Rightarrow \pi_1(B) = \pi_1(S^1\times P) \underset{\uparrow S^1 \,\&\, P \text{ path-conn.}}{=} \pi_1(S^1)\times\pi_1(P)$

$P \simeq \mathbb{R}^2\setminus\{0\} \underset{\uparrow\text{as above in case of } \mathbb{R}^3}{\simeq} S^1$

$\Rightarrow \pi_1(\mathbb{R}^3\setminus(\gamma\cup r)) = \mathbb{Z}\times\mathbb{Z} \underset{\text{iso}}{\not\cong} \mathbb{Z} = \pi_1(\mathbb{R}^3\setminus\gamma) \Rightarrow$ Spaces cannot be homeomorphic $\boxtimes$