Lemma 1: \( A = \{0,1,3\} \times \{0,2\} \cup [0,1] \times \{0,1,2,3\} \subseteq \mathbb{R}^2 \) has fundamental group \( \mathbb{Z} \ast \mathbb{Z} \).

Proof: Define \( A_1 = \{(x,y) \in A \mid y < 3\frac{3}{2}\} \), \( A_2 = \{(x,y) \in A \mid y > 3\frac{1}{2}\} \). We00000000\x2013;see that \( A = A_1 \cup A_2 \) and \( A_1 \) and \( A_2 \) are open in \( A \), and \( A_1 \cap A_2 \) contractible, therefore we know that \( \pi_1(A) \cong \pi_1(A_1) \ast \pi_1(A_2) \). \( A_1 \) is homotopic to a square since the leftovers on the corners are contractible. Since the square is homotopic to a circle we know that \( \pi_1(A_1) \cong \pi_1(S^1) \cong \mathbb{Z} \). And because \( A_1 \cong A_2 \Rightarrow \pi_1(A) \cong \mathbb{Z} \ast \mathbb{Z} \).

Lemma 2: \( B = \{0,1,3\} \times [2,3] \cup [0,1] \times 3 \frac{3}{2} \subseteq \mathbb{R}^2 \) has fundamental group \( \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \).

Proof: Define \( B_1 = \{(x,y) \in B \mid y < 3\frac{3}{2}\} \), \( B_2 = \{(x,y) \in B \mid y > 3\frac{3}{2}\} \). We can now apply the same theorem we used for Lemma 1. Since \( B_1 \cap B_2 \cong A_1 \cap A_2 \) and \( B_1 \cong A_1 \), \( B_2 \cong A_2 \Rightarrow \pi_1(B) \cong \pi_1(B_1) \ast \pi_1(B_2) \cong (\mathbb{Z} \ast \mathbb{Z}) \ast \mathbb{Z} \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \).

We will now use the same theorem we used in Lemma 1 and 2 and apply it to \( X \). Define \( X_4 = X \setminus \{0,1,3\} \times [0,1,3,4] \).

We can see that \( X_4 \) and \( X_2 \) are open and that their intersection is contractible. And clearly, \( X_2 \cong A \) and \( X_4 \cong B \). with \( X_4 \cap X_2 = X \Rightarrow \pi_1(X) \cong \pi_1(X_4) \ast \pi_1(X_2) \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \).

For the case where we remove a single point from \( X \) we will first look at the case where the point is not a corner. Then we can assume wlog that we remove the point \( (0,0,1/4) \) because of homotopy. Now we define \( X_3 = X_2 \setminus \{(0,1,3) \times [0,1,4] \} \) By removing this piece from \( X_2 \) we destroy one of the loops that means \( X_3 \) is now homotopic to \( S^4 \) and since \( X_4 \cap X_3 \) is still simply connected and \( X_3 \) open \( \Rightarrow \pi_4(X_3 \cap (0,0,1/4)) \cong \pi_4(X_3) \ast \pi_4(X_2) \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \).

In the case where we remove a corner, we will wlog remove \( (0,4,0) \) and can now retract the edges that were connected by this corner via homotopy until they have length \( 1/4 \) and we call the set \( X \). We define \( X_4 = X_4 \setminus \{(0,1,3) \times [0,1,4] \times [0,3/4] \} \) which is open and see that \( X = X_4 \cup X_2 \) and that \( X_4 \cap X_2 \) is simply connected, since \( X_4 \) is just the square on top with contractible leftovers at the corners \( \Rightarrow X_4 \cong S^4 \Rightarrow \pi_4(X) \cong \pi_4(X_4) \ast \pi_4(X_2) \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \).