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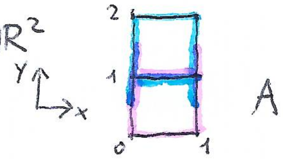
Topology ex. 10.10



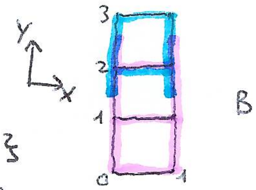
<u>Lemma 1</u>: $A = \{0,1\} \times [0,2] \cup [0,1] \times \{0,1,2\} \subseteq \mathbb{R}^2$ has fundamental group $\mathbb{Z} * \mathbb{Z}$

<u>Proof</u>: Define $A_1 := \{(x,y) \in A \mid y < 3/2\}$, $A_2 := \{(x,y) \in A \mid y > 1/2\}$

we see that $A = A_1 \cup A_2$, A_1 and A_2 are open in A, and $A_1 \cap A_2$ contractible, therefore we know that $\pi_1(A) \cong \pi_1(A_1) * \pi_1(A_2)$

A_1 is homotopic to a square, since the leftovers on the corners are contractible. Since the square is homotopic to a circle we know that $\pi_1(A_1) \cong \pi_1(S^1) \cong \mathbb{Z}$

And because $A_1 \simeq A_2 \implies \pi_1(A) \cong \mathbb{Z} * \mathbb{Z}$



<u>Lemma 2</u>: $B = A \cup \{0,1\} \times [2,3] \cup [0,1] \times \{3\} \subseteq \mathbb{R}^2$ has fundamental group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

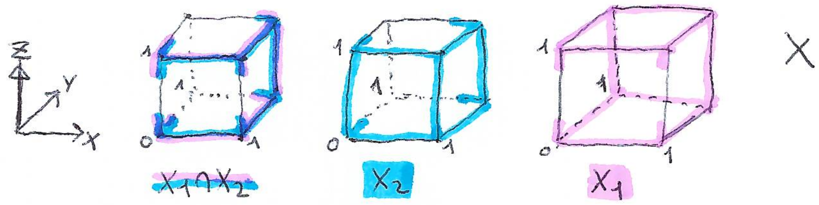
<u>proof</u>: Define $B_1 := \{(x,y) \in B \mid y < 5/2\}$, $B_2 := \{(x,y) \in B \mid y > 3/2\}$

we can now apply the same theorem we used for Lemma 1 since $B_1 \cap B_2 \simeq A_1 \cap A_2$, $B_1 \simeq A$, $B_2 \simeq A_1$

$\implies \pi_1(B) \cong \pi_1(B_1) * \pi_1(B_2) \cong (\mathbb{Z} * \mathbb{Z}) * \mathbb{Z} \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

We will know use the same theorem we used in Lemma 1 and 2 and apply it to X. Define $X_1 := X \setminus (\{0,1\} \times \{0\} \times [1/4, 3/4])$

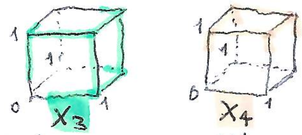
$X_2 := \{(x,y,z) \in X \mid y < 1/4 + x\}$



We can see that X_1 and X_2 are open and that their intersection is contractible.

And clearly, $X_2 \simeq A$ and $X_1 \simeq B$ with $X_1 \cup X_2 = X$

$\implies \pi_1(X) \cong \pi_1(X_1) * \pi_1(X_2) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$



For the case where we remove a single point from X we will first look at the case where the point is not a corner. Then we can assume wlog that we remove the point $(0,0,1/4)$ because of homotopy. Now we define $X_3 := X_2 \setminus ([0,1/4] \times [0,1/4] \times [0,1/4])$

By removing this piece from X_2 we destroy one of the Loops, that means X_3 is now homotopic to S^1 and since $X_1 \cap X_3$ is still simply connected and X_3 open $\implies \pi_1(X \setminus \{(0,0,1/4)\}) \cong \pi_1(X_1) * \pi_1(X_3) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

In the case where we remove a corner, we will wlog remove $(0,1,0)$. we can now retract the edges that were connected by this corner via homotopy until they have length $1/4$ and we call the set $\tilde{X}$.

we define $X_4 := X_1 \setminus ([0,1] \times [0,1] \times [0, 3/4])$ which is open and see that $\tilde{X} = X_4 \cup X_2$ and that $X_4 \cap X_2$ is simply connected, since X_4 is just the square on top with contractible leftovers at the corners $\implies X_4 \simeq S^1$

$\implies \pi_1(X \setminus (0,1,0)) \cong \pi_1(\tilde{X}) \cong \pi_1(X_2) * \pi_1(X_4) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$