Topology Challenge Exercise 10 Kevin Zhang

One can project C onto $\mathbb{R}^2\times\{0\}$, where $x\in X$ is projected onto the intersection of $\mathbb{R}^2\setminus\{0\}$ and $(1/2,1/2,-1/2)+\langle x-(1/2,1/2,-1/2)\rangle$. This projection is continuous and a bijection between X and it's image Y, since for any $x\in X$, $(1/2,1/2,-1/2)+\langle x-(1/2,1/2,-1/2)\rangle\cap X=\{x\}$. Furthermore since X and Y are compact and hausdorff, X and Y are homeomorphic.

Like you can see below, for Y as a subset of $\mathbb{R}^2$, $\mathbb{R}^2\setminus\{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{6}),(\frac{1}{2},\frac{5}{6}),(\frac{1}{6},\frac{1}{2}),(\frac{5}{6},\frac{1}{2})\}$ deformation retracts to Y. Thus by 10.8 we get $\pi_1(X)\cong\pi_1(Y)\cong\pi_1(\bigwedge_{i=1}^{5}\mathbb{Z})\cong\ast_{i=1}^{5}\mathbb{Z}$.

Now consider any $x\in X$. Suppose x is a corner. W.l.o.g. $x=(0,0,0)$, the other cases are symmetric. Then $X\setminus\{x\}$ is homeomorph to $Y\setminus(0,0,0)$. Now $\mathbb{R}^2\setminus\{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{5}{6}),(\frac{5}{6},\frac{1}{2})\}$ deformation retracts to $Y\setminus(0,0,0)$, thus in this case $\pi_1(X\setminus\{x\})\cong\pi_1(\bigwedge_{i=1}^{3}\mathbb{Z})\cong\ast_{i=1}^{3}\mathbb{Z}$.

Now for any other x, w.l.o.g we can assume $x\in(0,1)\times\{0\}\times\{0\}$. Then $X\setminus\{x\}$ is homeomorph to $Y\setminus\{y\}$ for some $y\in(0,1)\times\{0\}$. Now $\mathbb{R}^2\setminus\{(\frac{1}{2},\frac{1}{2}),(\frac{1}{6},\frac{1}{2}),(\frac{1}{2},\frac{5}{6}),(\frac{5}{6},\frac{1}{2})\}$ deformation retracts to $Y\setminus\{y\}$, thus in this case $\pi_1(X\setminus\{x\})\cong\pi_1(\bigwedge_{i=1}^{4}\mathbb{Z})\cong\ast_{i=1}^{4}\mathbb{Z}$.