<u>11.10:</u> Consider $\mathbb{C}^2$ as $\underset{\Phi}{\_}\ \mathbb{R}^4$

<u>ie.</u> $(a+bi, c+di) \overset{\Phi}{\simeq} (a, b, c, d)$
where $a, b, c, d \in \mathbb{R}$.

Note that $\Phi(l_1), \Phi(l_2)$ are disjoint planes (except for 0) in $\mathbb{R}^4$ because for some $\begin{pmatrix} a_1+b_1 i \\ c_1+d_1 i \end{pmatrix}, \begin{pmatrix} a_2+b_2 i \\ c_2+d_2 i \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\}$

$$l_1 = \left\{ \lambda \begin{pmatrix} a_1+b_1 i \\ c_1+d_1 i \end{pmatrix} \;\middle|\; \lambda \in \mathbb{C} \right\}$$

$\lambda = \lambda_1 + \lambda_2 i$

$$= \left\{ \lambda_1 \begin{pmatrix} a_1+b_1 i \\ c_1+d_1 i \end{pmatrix} + \lambda_2 \begin{pmatrix} -b_1 + a_1 i \\ -d_1 + c_1 i \end{pmatrix} \;\middle|\; \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{R}^2 \right\}$$

$$\simeq \left\langle \underbrace{\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}}_{x :=}, \underbrace{\begin{pmatrix} -b_1 \\ a_1 \\ -d_1 \\ c_1 \end{pmatrix}}_{=: \tilde{x}} \right\rangle .$$

where $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} -b_1 \\ a_1 \\ -d_1 \\ c_1 \end{pmatrix}$ linearly independent.

and similarly

$$l_2 \simeq \left\langle \overbrace{\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}}^{=: y}, \overbrace{\begin{pmatrix} -a_2 \\ b_2 \\ -d_2 \\ c_2 \end{pmatrix}}^{=: \tilde{y}} \right\rangle .$$

Assume non-trivial intersection of $v \in \Phi(l_1) \cap \Phi(l_2)$

i.e. $\Phi(v) \in l_1 \cap l_2$: then $\Phi^{-1}(v) = \begin{pmatrix} v_1 + v_2 i \\ v_3 + v_4 i \end{pmatrix}$

for $v_1, v_2, v_3, v_4 \in \mathbb{R}$.

Then $l_1 = \langle v \rangle = l_2$ ↯ (l_1, l_2 are distinct.

Thus $\{x, \tilde{x}, y, \tilde{y}\}$ is a basis of $\mathbb{R}^4$.

Consider the cont. change of coordinates:

$$\mathbb{R}^4 \xrightarrow{\sim} \mathbb{R}^4$$

$$e_1 \longmapsto x \ , \ e_2 \mapsto \tilde{x} \ , \ e_3 \mapsto y \ , \ e_4 \mapsto \tilde{y}$$

Such a map is clearly Lipschitz cont. and bijective with Lipschitz cont. inverse and thus a homeomorphism.

Note that we have reduced the problem to computing the fundamental group of $\mathbb{R}^4 \setminus (\mathbb{R}^2 \times \{0\}^2 \cup \{0\}^2 \times \mathbb{R}^2)$

$= \mathbb{R}^2 \times (\mathbb{R}^2)^* \setminus \{0\} \times \mathbb{R}^2 = (\mathbb{R}^2)^* \times (\mathbb{R}^2)^*$

$\cong (S^1 \times \mathbb{R}^+) \times (S^1 \times \mathbb{R}^+)$ where $(\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{\binom{0}{0}\}$

polar coordinates

But note that

$$H : X \times I \longrightarrow X$$

$$H(x,t) = (\alpha, r - t(r-1), \beta, s - t(s-1))$$

$$x = (\underset{S^1}{\alpha} , \underset{\mathbb{R}^+}{r} , \underset{S^1}{\beta} , \underset{\mathbb{R}^+}{s})$$

is a well-defined deformation retraction onto

$$S^1 \times \{1\} \times S^1 \times \{1\} \cong S^1 \times S^1$$

homeomorphic

$$\Longrightarrow \pi_1(X) \cong \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$$

$$\pi_1(X) = \pi_1(\mathbb{C}^2 \setminus (\ell_1 \cup \ell_2))$$

$$= \mathbb{Z} \times \mathbb{Z}$$