Topology Challenge Exercise 11 Kevin Zhang

W.l.o.g. we may assume $X = S^2 \dot\cup (\{0\}\times\{0\}\times(-1,1))$. Then let $\tilde{X} := \dot\bigcup_{k\in\mathbb{Z}} ((S^2 \dot\cup \{0\}\times\{0\}\times(1,3)) + (0,0,4k))$ and let $p: \tilde{X} \to X$ be given by $x + (0,0,4k) \mapsto x$ for $x \in S^2 + (0,0,4k)$ and $x + (0,0,4k) \mapsto -(x-(0,0,2))$ for $x \in \{0\}\times\{0\}\times[1,3] + (0,0,4k)$ for $k\in\mathbb{Z}$. One can quickly check that this is a cover map, and since $\tilde{X}$ is simply connected, this is an universal cover.

For the other case, w.l.o.g. X is given by $S^2 \cup \{(0,x,y) \mid (x-1)^2+y^2=0\}$.

We can construct a cover space $\tilde{X}$ as follows

Start with $A := S^2 \cup \{0\}\times\{0\}\times[1,3] \cup \{0\}\times[0,2]\times\{1\} \cup \{0\}\times\{0\}\times[-3,-1] \cup \{0\}\times[-2,0]\times\{1\}$.

Then inductively add A scaled down by 3 to the endpoints of the lines, s.t. the four lines above are tangents, and the lines only connect north and south poles.

E.g. after one step we have $A \cup (\frac{A}{3} + (0,0,3\frac{1}{3})) \cup (\frac{A}{3} + (0,2,1\frac{1}{3})) \cup (\frac{A}{3} + (0,0,-3\frac{1}{3})) \cup (\frac{A}{3} + (0,-2,-1\frac{1}{3}))$.

A sketch of the intersection with $\{0\}\times\mathbb{R}^2$ can seen below.

A cover map can then be defined by mapping the spheres in $\tilde{X}$ by stretching and a translation onto S^2. The vertical lines are homeomorphically mapped onto the arc of the circle in the interior of S^2, and the horizontal lines onto the arc on the exterior, whereas the orientations are given as below. Since $\tilde{X}$ is also simply connected, we have found a simply connected cover.