1. Topological spaces and continuous functions

1.1. Question with four points ✍. Let $X = \{a, b, c, d\}$ a set with four points. Which of the following ones are topologies for X?

(i) $\{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$

(ii) $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$

(iii) $\{\emptyset, X, \{a, c, d\}, \{b, c, d\}\}$

1.2. The arrow line ✍. For each $x \in \mathbb{R}$, let $I_x = (x, \infty)$, and let $I_\infty = \emptyset$ and $I_{-\infty} = \mathbb{R}$. Check that

$$\mathcal{T} = \{I_x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}\}$$

defines a topology on $\mathbb{R}$.

1.3. Point and finite complements ⚙. Let X be a set and let p be an element of X. Check that

$$\mathcal{T} = \{A \subseteq X \mid p \notin A \text{ or } X \setminus A \text{ is finite}\}$$

defines a topology on X.

1.4. Continuous function ✍. Let $\mathcal{T}$ be the topology for $\mathbb{R}$ described in Problem 1.2. Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous *with respect to* $\mathcal{T}$?

(i) $f(x) = x^2$

(ii) $f(x) = x^3$

(iii) $f(x) = \begin{cases} 5 & \text{if } x > 5 \\ 0 & \text{otherwise} \end{cases}$

(iv) $f(x) = -x$

1.5. Elaborate on Problem 1.4 ⚙. In this exercise, we want to understand a little bit better continuous maps in the topology of Problem 1.2. For the sake of this exercise, we say that a map $f : \mathbb{R} \to \mathbb{R}$ is *standard-continuous*, if it is continuous with respect to the usual topology on $\mathbb{R}$. We say that it is $\mathcal{T}$*-continuous* if it is continuous with respect to the topology described in Problem 1.2. Let f be a function that is standard-continuous. Prove that f is $\mathcal{T}$-continuous if and only if it is monotonically increasing.

1.6. Definitions of interior ✍. The goal of this exercise is to give some equivalent characterizations for the *interior* of a set. Let X be a topological space and let Y be a subset of X. Moreover, let:

(i) $\text{int}(Y) = \{x \in X \mid \text{ there exists } O \text{ open such that } x \in O \subseteq Y\}$;

(ii) Y_1 be the maximal open set that is contained in Y (if it exists);

(iii) Y_2 be the union of all the open sets that are contained in Y.

Show that Y_1 exists and $\mathrm{int}(Y) = Y_1 = Y_2$.

1.7. Definitions of closure. The goal of this exercise is to give some equivalent characterizations for the *closure* of a set. Let X be a topological space and let Y be a subset of X. Let:

(i) $\overline{Y} = \mathrm{int}(Y) \cup \{x \in X \mid \text{ for each open } O \ni x,\, O \cap Y \neq \emptyset \neq O \cap (X \setminus Y)\}$;

(ii) Y_1 be the minimal closed set that contains Y (if it exists);

(iii) Y_2 be the intersection of all the closed sets that contain Y;

(iv) $Y_3 = X \setminus \mathrm{int}(X \setminus Y)$.

Show that Y_1 exists and $\overline{Y} = Y_1 = Y_2 = Y_3$.

1.8. Examples with closure. Give an example of two subsets A and B of $\mathbb{R}$ such that:

$$A \cap B = \emptyset, \qquad \overline{A} \cap B \neq \emptyset, \qquad A \cap \overline{B} \neq \emptyset.$$

Bonus: can you find two (essentially different) such examples?

1.9. Union and closure. Let A and B be subsets of a topological space X. Show that:

(i) $\mathrm{int}(A) \cap \mathrm{int}(B) = \mathrm{int}(A \cap B)$;

(ii) $\mathrm{int}(A) \cup \mathrm{int}(B) \subseteq \mathrm{int}(A \cup B)$;

(iii) $\overline{A} \cup \overline{B} = \overline{A \cup B}$;

(iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;

(v) Give one example where the equality in part (ii) is satisfied, one where it fails, one where the equality in part (iv) is satisfied and one where it fails.

1.10. Topology with many open sets. Prove that a topology on $\{1, 2, \ldots, n\}$ with k open sets cannot exist if $3 \cdot 2^{n-2} < k < 2^n$.