2. Bases, subspaces and products

2.1. Homeomorphisms. Show that the following topological spaces are homeomorphic.

(i) The interval $[0,1]$ and the interval $[2,5]$.

(ii) The interval $(-1,1)$ and the real line $\mathbb{R}$.

(iii) The closed disk of radius one in $\mathbb{R}^2$ and the closed square $[-1,1]\times[-1,1]$ in $\mathbb{R}^2$.

2.2. Existence of infinitely many primes. Let $\mathbb{Z}$ be the set of integer numbers. For every pair of integers $a,b\in\mathbb{Z}$ with $b>0$, let $B_{a,b}$ be the set

$$B_{a,b} \coloneqq \{a+kb \,:\, k\in\mathbb{Z}\}.$$

Prove the following facts:

(i) The set $\mathcal{B}=\{B_{a,b} \,:\, a,b\in\mathbb{Z}, b>0\}$ forms a basis for a topology $\mathcal{T}$ on $\mathbb{Z}$.

(ii) For every $a,b\in\mathbb{Z}$ with $b>0$, the set $B_{a,b}$ is both open and closed in $\mathbb{Z}$ with respect to $\mathcal{B}$.

(iii) Let $P=\{2,3,5,\dots\}$ be the set of prime numbers. Use the above facts to show that P needs to be infinite. *Hint: Consider the set* $\mathbb{Z}\setminus\bigcup_{p\in P}B_{0,p}$.

2.3. Non-standard topology. Let $\mathcal{B}$ be the following family of subsets of $\mathbb{R}$:

$$\mathcal{B}\coloneqq\{[a,b) \,:\, a\in\mathbb{Z}, b\in\mathbb{R}, a<b\}.$$

(i) Prove that there exists a topology $\mathcal{T}$ for which $\mathcal{B}$ is a basis.

(ii) Determine the interior and the closure of $A=(1/2,2)$ in the topology $\mathcal{T}$.

2.4. Product of maps. Let X,Y,Z,W be topological spaces. Given two functions $f\colon X\to Z$ and $g\colon Y\to W$, we can define their product map $(f\times g)\colon X\times Y\to Z\times W$ as $(f\times g)(x,y)\coloneqq(f(x),g(y))$, which is continuous if and only if f and g are continuous.

(i) Show that if f and g are open then so is $f\times g$.

(ii) Show with a counterexample that the product of closed functions is not necessarily closed.

Note: We say that a function $f\colon X\to Z$ is open *if for every open set $O\subseteq X$ we have that $f(O)$ is open. Similarly f is* closed *if the image of each closed set is closed.*

2.5. Interior and closure of a product. Let X and Y be topological spaces, and let A,B be subsets of X,Y respectively. Show that

(i) $\operatorname{int}(A)\times\operatorname{int}(B)=\operatorname{int}(A\times B)$.

(ii) $\overline{A} \times \overline{B} = \overline{A \times B}$.

2.6. Sub-subspaces ✍. Let X be a topological space equipped with a topology $\mathcal{T}_X$, and let Y be a subset of X with the subset topology $\mathcal{T}_Y$ induced by $\mathcal{T}_X$. Given a subset Z of Y, we can consider on it the subset topologies $\mathcal{T}_{Z,Y}$ and $\mathcal{T}_{Z,X}$ induced by $\mathcal{T}_Y$ and $\mathcal{T}_X$, respectively. Show that $\mathcal{T}_{Z,Y} = \mathcal{T}_{Z,X}$.

2.7. Interior of subspaces ✍. Let Y be a subspace of a topological space X (i.e. Y is a topological space equipped with the subspace topology) and let A be a subset of Y. Defining $\operatorname{int}_X(A)$, $\operatorname{int}_Y(A)$ as the interiors of A with respect to X and Y respectively, show that $\operatorname{int}_X(A) \subseteq \operatorname{int}_Y(A)$ and give an example where the equality does not hold.

2.8. Finite metric space ✍. Let (X, d) be a metric space consisting of a finite number of points. Show that in X the distance topology coincides with the discrete topology.

2.9. p-adic numbers ⚙. Let p be a prime number and $d_p\colon \mathbb{Z} \times \mathbb{Z} \to [0, \infty)$ be the function defined by

$$d_p(x, y) := p^{-\max\{m \in \mathbb{N} \,:\, p^m \mid x-y\}},$$

where we understand $p^{-\max\{m \in \mathbb{N}\}}$ to be 0. Prove that d_p is a metric on $\mathbb{Z}$ and that $d_p(x, y) \leq \max\{d_p(x, z), d_p(z, y)\}$ for every $x, y, z \in \mathbb{Z}$.

Note: Here $p^m \mid x - y$ means that p^m divides $x - y$.

2.10. Open sets with the same boundary ⬡. For which $n, k \in \mathbb{N}_* = \{1, 2, 3, \ldots\}$ do there exist k distinct open subsets of $\mathbb{R}^n$ with the same boundary?