3. Connectedness

3.1. Connected union of disks. Let $p = (-1, 0)$ and $q = (2, 0)$ be points in $\mathbb{R}^2$, and let $D_1 = \{z \in \mathbb{R}^2 : |z - p| < 1\}$ and $D_2 = \{z \in \mathbb{R}^2 : |z - q| < 2\}$. Which of the following subsets of $\mathbb{R}^2$ are connected?

(i) $D_1 \cup D_2$

(ii) $\overline{D_1} \cup D_2$

(iii) $\overline{D_1} \cup \overline{D_2}$

Note: You may use the fact that $\overline{D_1} = \{z \in \mathbb{R}^2 : |z - p| \leq 1\}$ *and* $\overline{D_2} = \{z \in \mathbb{R}^2 : |z - q| \leq 2\}$.

3.2. Connectedness in discrete topology. Let X be a set equipped with the discrete topology. Which subsets of X are connected?

3.3. Product arc connected. Let X and Y be non-empty topological spaces. Show that $X \times Y$ is path-connected if and only if both X, Y are path-connected.

3.4. From the circle to the real line. Let $S^1 = \{z \in \mathbb{R}^2 : |z - (0, 0)| = 1\}$ be the unit circle in $\mathbb{R}^2$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. Show that there exists $z \in S^1$ such that $f(z) = f(-z)$. In particular, f is not injective.

Note: Given $z = (x_0, y_0) \in S^1$, *we denote by* $-z$ *the point* $(-x_0, -y_0) \in S^1$.

3.5. Closure and connectedness. Let X be a topological space and let A be a subset of X. Show that, if B is a subset of X with $A \subseteq B \subseteq \overline{A}$ and A is connected, then so is B.

3.6. Connectedness in the real line. Show that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

3.7. Totally disconnected in the real line. Show that a subset of $\mathbb{R}$ is totally disconnected if and only if it does not contain any non-empty open interval.

Note: A topological space X *is* totally disconnected *if every connected subspace* $A \subseteq X$ *either is the empty set or consists of a single element.*

3.8. Complement of a countable set. If A is countable then $\mathbb{R}^2 \setminus A$ is path-connected.

3.9. Homeo(and diffeo)morphisms between Euclidean spaces. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ and that $\mathbb{R}^p$ is not diffeomorphic to $\mathbb{R}^q$ for any $p \neq q$.

Note: In fact it is true that $\mathbb{R}^p$ *is not homeomorphic to* $\mathbb{R}^q$ *for any* $p \neq q$ *but the proof is much more involved. Later on in the course we will show the result in the case* $p = 2$ *and* $q \geq 3$*, while the proof of the general result needs finer algebraic topology tools.*

3.10. Special subsets of the plane. Let G be a nonempty subset of $\mathbb{R}^2$ that is closed under addition, symmetric with respect to the origin and path-connected. Prove that G is a linear subspace of $\mathbb{R}^2$.

Note: The following theorem may be useful.

Jordan curve theorem. *Let* $C \subseteq \mathbb{R}^2$ *be a simple closed curve (i.e.* C *is the image of an injective continuous map* $\gamma : S^1 \to \mathbb{R}^2$*). Then* $\mathbb{R}^2 \setminus C$ *consists of exactly two connected components.*