4. Compactness

Chef's table

Starting this week, based on some requests I have received, I would like to add a few comments about the problem set, so to guide those among you who wish to focus on just five or six of them. As a general remark, even if you decide to write down accurate solutions only to a certain subset of exercises, it may be a great idea to take some time to think about all of them (with the exception of the challenge problem, if you have time constraints). Meditating on a problem without a piece of paper and a pen in front you is very good (and helpful) practice, which reinforces your abstraction skills and gets you closer to a research-type experience. It may be difficult at first, but you should give it a try.

That said, I envision two types of tasting menus. A *lighter option* might be **4.1 - 4.2 - 4.3 - 4.4 - 4.5 - 4.6**: these exercises are all pretty short, but they provide a good training. In particular, Problem 4.5 is very instructive. A *more demanding option* could be **4.5 - 4.7 - 4.8 - 4.9**: the combination of 4.8 and 4.9 provides a complete proof for a (very important!) characterisation of compact sets in Euclidean spaces, and the statement should be known to all students in the class. Overall, this couple of problems is rather lengthy to be written down in detail, but provides an excellent practice both from the technical viewpoint and from a *writing in the Major* perspective. Finally, let me note that Problem 4.6 is a very basic but helpful result in Real Analysis: for instance, it makes it much simpler to prove that the set of limit points of the sequence $a_n = \sin(n)$ coincides with the closed interval $[-1, 1]$, which would be a lot harder to prove with purely elementary tools.

4.1. Discrete topology ✎. Let X be a set equipped with the discrete topology. Characterize the compact subspaces of X.

4.2. Finite intersection property ✎. We say that a family $\mathcal{A}$ of subsets of a topological space X has the *finite intersection property* if for each (non-empty) finite subfamily $\mathcal{F}$ of $\mathcal{A}$ we have that

$$\bigcap_{A\in\mathcal{F}} A \neq \emptyset.$$

Show that a topological space X is compact if and only if, for every family of closed subsets $\mathcal{A}$ that has the finite intersection property, we have that

$$\bigcap_{A\in\mathcal{A}} A \neq \emptyset.$$

4.3. Intersection of compact sets ✎. Let X be a compact topological space, O be an open subset of X an $\{C_i\}_{i\in I}$ be a (possibly infinite) family of closed sets such that

$$\bigcap_{i\in I} C_i \subseteq O.$$

Show that it is possible to find a *finite* set of indices $\{i_1, \dots, i_n\} \subseteq I$ such that

$$\bigcap_{k=1}^{n} C_{i_k} \subseteq O.$$

4.4. Finite number of digits. Given a topological space X, let X' be the subspace of X obtained by removing all the isolated points of X, i.e. all the points of X which are open and closed in X. Let B_n be the subspace of $[0, 1]$ that consists of all the numbers having a base 2 decimal expansion $0.a_1a_2a_3\dots$ in which at most n of the digits a_i are 1, and let $B := \cup_{n\in\mathbb{N}} B_n$. Determine B' and B'_n for every $n \in \mathbb{N}$. Deduce that there for each $n \in \mathbb{N}$ there is a space X such that the sequence

$$X \supseteq X' \supseteq X'' \supseteq \dots$$

becomes the empty set after exactly n stages.

4.5. The cofinite topology. Let $\mathcal{T}$ be the family of subsets of the real line $\mathbb{R}$ defined as

$$\mathcal{T} := \emptyset \cup \{\mathbb{R} \setminus F \,:\, F \subset \mathbb{R} \text{ is finite}\}.$$

(i) Check that $\mathcal{T}$ is a topology and that $(\mathbb{R}, \mathcal{T})$ is compact.

(ii) Let $\mathcal{T}_{\text{std}}$ be the standard topology on $\mathbb{R}$. Show that $(\mathbb{R}, \mathcal{T})$ and $(\mathbb{R}, \mathcal{T}_{\text{std}})$ are not homeomorphic.

4.6. Limit points of a sequence. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence of points in $\mathbb{R}$. We say that $y \in \mathbb{R}$ is a *limit point* for this sequence if there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ that converges to y. Prove that the set of the limit points of any sequence in $\mathbb{R}$ is closed.

4.7. Neighborhood of a set. Let C be a closed subset of $\mathbb{R}^n$ and let A be an open subset of $\mathbb{R}^n$ that contains C. For every $\varepsilon > 0$, define $C_\varepsilon := \{x \in \mathbb{R}^n \,:\, d(x, C) < \varepsilon\}$. Prove that, if C is compact, then there exists $\varepsilon > 0$ such that $C_\varepsilon \subseteq A$. Is the conclusion true removing the hypothesis of C being compact?

Note: Given any $x \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$, the distance of x to S is defined as $d(x, S) := \inf_{s\in S}|x - s|$.

4.8. Preparation to Problem 4.9. Before facing Problem 4.9, we need the following preliminary facts.

(i) Let (X, d) be a metric space and assume that X is *separable*, which means that X contains a countable dense subset. Prove that any open cover $\mathcal{O}$ of X admits a countable subcover.

(ii) Let (X, d) be a complete metric space. Then a subset $Y \subseteq X$ is closed if and only if it is complete. Observe that this applies in particular to $X = \mathbb{R}^n$ with the Euclidean distance.

(iii) Prove that a subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded. Show that this is not true in general in a complete metric space (X, d).

4.9. Equivalent notions of compactness ⚙. Given a metric space (X, d), show that the following conditions are equivalent:

(C1) The space X is compact (i.e. every open cover of X admits a finite subcover).

(C2) The space X is sequentially compact (i.e. every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ admits a converging subsequence).

(C3) The space X is complete (i.e. every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$) and totally bounded (i.e. for every $\varepsilon > 0$ there exists a finite set of points $x_1, \ldots, x_k \in X$ such that $X \subseteq \cup_{i=1}^k B(x_i, \varepsilon)$).

Note: Observe that, thanks to (ii) *and* (iii) *in Problem 4.8, this statement is equivalent to the analogous one seen in class in the case when* (X, d) *is a subspace of a Euclidean space.*

4.10. Union of strictly convex compact sets. Prove that it is not possible to obtain $\mathbb{R}^n$ as a countable union of strictly convex compact sets that are pairwise disjoint.