# 5. Separation axioms and related stories

### Chef's table

This week, the problem set we present is a lot shorter than usual (our solutions are less than two pages altogether). If you set aside the challenge problem, the other nine exercises could be defined as *quick review questions* and provide a good idea of the sort of tasks you will face in Section I.b of the exam. So you may all want to go for the complete tasting menu this week.

Among these problems, 5.7-5.8-5.9 concern separable spaces (namely: topological spaces containing a countable dense subset). The first two exercises are key facts that are often employed in Functional Analysis (stay tuned...). Instead, Problem 5.9 indicates what is perhaps the simplest example of a non-separable metric (in fact: Banach) space, and all students should have it in mind.

Lastly, the challenge problem this week (which is perhaps a bit more accessible than usual) shows that the equivalence we proved in Problem 4.9 last week does not hold as we move outside of the kingdom of metric spaces.

**Remind:** A topological space $X$ is said to be *first-countable* if each point has a countable basis of neighborhoods, i.e. for every point $x \in X$ there exists a countable family $\mathcal{B}$ of open neighborhoods of $x$ such that for every open set $U$ that contains $x$ there is $B \in \mathcal{B}$ with $B \subseteq U$. On the other hand, we say that a topological space is *second-countable* if it admits a countable basis for its topology.

**5.1. Hausdorff and infinite products.** Let $\{X_i\}_{i\in I}$ be a family of Hausdorff topological spaces. Show that $X = \prod_{i\in I} X_i$ is a Hausdorff space.

**5.2. Example of non-Hausdorff space.** Give an example of a topological space that is not Hausdorff.

**5.3. Continuous bijection and Hausdorff.** Let $X$ and $Y$ be topological spaces. Suppose that $Y$ is Hausdorff and that there is a continuous bijection $f\colon X \to Y$. Show that $X$ is Hausdorff.

**5.4. Subsets of Hausdorff spaces are Hausdorff.** Let $X$ be a Hausdorff topological space and let $Y$ be a subset of $X$. Show that $Y$ is Hausdorff with respect to the induced topology.

**5.5. Convergence and basis.** Let $X$ be a first-countable topological space, $x$ be a point of $X$ and $\{O_k\}_{k\in\mathbb{N}}$ be a basis of neighborhoods for $x$.

(i) For every $n \in \mathbb{N}$, let $U_n := \bigcap_{k=1}^{n} O_k$, and let $x_n$ be any point in $U_n$. Show that $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$.

(ii) Let $\{y_i\}_{i\in\mathbb{N}}$ be a sequence such that for every $n, k \in \mathbb{N}$ there is $i > n$ such that $y_i \in O_k$. Show that there exists a subsequence of $\{y_i\}_{i\in\mathbb{N}}$ that converges to $x$.

**5.6. Non-first-countable topology on the real line ⚙.** Let $\mathcal{T}$ be the family of subsets of the real line $\mathbb{R}$ defined in Problem 4.5, that is

$$\mathcal{T} := \emptyset \cup \{\mathbb{R} \setminus F \,:\, F \subset \mathbb{R} \text{ is finite}\}.$$

Show that $(\mathbb{R}, \mathcal{T})$ is not first-countable.

**5.7. Separable and first-countable space ⚙.** Let $X$ be a separable first-countable topological space. Show that any dense subspace $A$ of $X$ is separable. Check that a metric space is first-countable, hence every dense subspace of a separable metric space is separable.

**5.8. Separability and product ✎.** Given two topological spaces $X$ and $Y$, show that $X \times Y$ is separable if and only if both $X$ and $Y$ are separable.

**5.9. $L^\infty$ is not separable ⚙.** Given a measurable function $f\colon [0,1] \to \mathbb{R}$ we define its *essential supremum* as

$$\operatorname{ess\,sup} |f| := \inf\{c \in [0,\infty] \,:\, |f| \le c \text{ almost everywhere}\}.$$

Consider the space

$$L^\infty([0,1]) := {\{f\colon [0,1] \to \mathbb{R} \,:\, f \text{ measurable}, \operatorname{ess\,sup}|f| < \infty\}}\big/{\sim},$$

where the relation $\sim$ is defined as $f \sim g$ if and only if $\operatorname{ess\,sup}|f - g| = 0$. Show that $L^\infty([0,1])$ equipped with the distance $d_\infty(f,g) := \operatorname{ess\,sup}|f-g|$, is not separable.

*Note: On the other hand, for any $p \in [1,\infty)$, the space $L^p([0,1])$ equipped with the distance function $d_p(f,g) := \|f-g\|_p$ is separable. Indeed, a countable dense subset is given by*

$$\left\{\sum_{i=k}^{n} a_k \chi_{I_k} \,:\, a_k \in \mathbb{Q},\ I_k \subseteq [0,1] \text{ open interval with end points in } \mathbb{Q}\right\}.$$

**5.10. Compact but not sequentially compact ⊛.** Let $I = [0,1]$ and consider the space $I^I$. Show that $I^I$ is compact but not sequentially compact.

*Note: The space $I^I$ is the product space of $I$-many copies of $I$, i.e. $\prod_{i\in I} I$, with the product topology.*