6. Metric spaces

Chef's table

In this problem set, we want to test all general topological notions we have so far acquired in the specific context of metric spaces. As you can see, important metric spaces naturally arise as *functional spaces* (i.e. spaces whose points are functions). In the set below, the most significant exercises are perhaps the first two, 6.1 and 6.2, that lead to a complete proof (at the greatest possible level of generality) of the Arzelà-Ascoli compactness theorem, which is arguably one of the most useful compactness results in Mathematics and the basis for a number of advanced results.

At a conceptual level, we also investigate the question of determining when a distance (on a vector space) is determined from a norm, see Problem 6.8, and give a first glance at the way finite-dimensional vector spaces differ from infinite-dimensional ones, which is the real theme behind problems 6.5 and 6.6.

So, in this problem set you see three category of objects:

(i) topological spaces

(ii) metric spaces

(iii) normed linear spaces

each class being, in a natural way, included in the previous one (with two *proper* inclusions). The results you prove here will come back again and again along your mathematical path.

6.1. Space of continuous functions. Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. Consider the space of continuous functions from X to Y, denoted by $C(X,Y) := \{f \colon X \to Y \,:\, f \text{ is continuous}\}$. We define a distance in $C(X,Y)$ as $d(f,g) := \sup_{x\in X} d_Y(f(x), g(x))$ for all $f, g \in C(X,Y)$.

(i) Show that d is indeed a metric on $C(X,Y)$.

(ii) Show that $C(X,Y)$ with distance d is a complete metric space.

6.2. Arzelà-Ascoli theorem. Consider a compact metric space (X, d_X) and a complete metric space (Y, d_Y). Then, as in Problem 6.1, consider the metric space $(C(X,Y), d)$ of continuous functions from X to Y. Prove that a subset $\mathcal{F} \subseteq C(X,Y)$ is *relatively compact* (i.e. $\overline{\mathcal{F}}$ is compact) if and only if it is

- *pointwise relatively compact*, i.e. $\mathcal{F}_x := \{f(x) \,:\, f \in \mathcal{F}\}$ is relatively compact in Y for all $x \in X$, and
- *equicontinuous*, i.e. for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d_X(x,y) < \delta$.

6.3. Two sequences in a metric space. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two Cauchy sequences in a metric space (X,d). Prove that the sequence $\{d(x_n,y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges.

6.4. Subset of complete is complete. Let X be a complete metric space and let Y be a subset of X. Show that Y is complete if and only if it is closed.

6.5. Norms in a finite-dimensional space. Given a vector space X over $\mathbb{R}$, we say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are *equivalent* if

$$\exists C>0: \quad C^{-1}\|x\|' \leq \|x\| \leq C\|x\|' \quad \forall x\in X.$$

Show that, if X is finite-dimensional, all norms on X are equivalent.

6.6. Non-equivalent distances and norms. Similarly to the case of norms (see Problem 6.5), we say that two metrics d and d' on a set X are *equivalent* if

$$\exists C>0: \quad C^{-1}d'(x_1,x_2) \leq d(x_1,x_2) \leq Cd'(x_1,x_2) \quad \forall x_1,x_2\in X.$$

(i) Construct two metrics on $\mathbb{R}^2$ that are *not* equivalent.

(ii) Construct a vector space X with two norms $\|\cdot\|$ and $\|\cdot\|'$ that are *not* equivalent.

Hint: Prove that $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent by exhibiting a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ that converges for $\|\cdot\|$ but not for $\|\cdot\|'$.

6.7. p-adic distance. Let p be a prime number. Prove that the p-adic distance $d_p\colon \mathbb{Z}\times\mathbb{Z}\to[0,\infty)$ defined in Problem 2.9 is not equivalent to the Euclidean distance $d(x,y) := |x-y|$.

6.8. Metric induced by norm. Let V be a vector space over $\mathbb{R}$. Show that a metric d on V is induced by a norm $\|\cdot\|$ (i.e. there exists a norm $\|\cdot\|$ such that $d(x,y)=\|x-y\|$ for all $x,y\in V$) if and only if the metric is

- *translation invariant*, i.e. $d(x+v,y+v)=d(x,y)$ for all $x,y\in V$ and $v\in V$, and
- *homogeneous*, i.e. $d(\lambda x,\lambda y)=|\lambda|d(x,y)$ for all $x,y\in V$ and $\lambda\in\mathbb{R}$.

6.9. A metric on $C^0(\mathbb{R}^m)$. Let $K_1\subset K_2\subset\ldots\subset\mathbb{R}^m$ be a family of compact subsets such that $K_n\subset\operatorname{int}(K_{n+1})$ for every $n\in\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}K_n=\mathbb{R}^m$.

(i) Prove that

$$d(f,g) := \sum_{n\in\mathbb{N}} \frac{2^{-n}\|f-g\|_{C^0(K_n)}}{1+\|f-g\|_{C^0(K_n)}}$$

defines a metric d on $C^0(\mathbb{R}^m)$.

(ii) Show that $(C^0(\mathbb{R}^m), d)$ is complete.

(iii) Show that $C^0_c(\mathbb{R}^m)$, the space of continuous functions with compact support, is dense in $(C^0(\mathbb{R}^m), d)$.

Note: The same conclusions with the same proofs also hold for any *open set* $\Omega \subseteq \mathbb{R}^m$ *in place of* $\mathbb{R}^m$*. The metric* d *deals with the fact that* $C^0(\mathbb{R}^m)$ *contains unbounded functions like* $f(x) = |x|^2$ *for which* $\sup_{x\in\mathbb{R}^m}|f(x)| = \infty$*.*

6.10. Expanding map on a compact metric space. Let (X, d) be a compact metric space and let $f : X \to X$ be an expanding map, i.e. $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Prove that f is an isometry, i.e. $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.